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The topological B-model on fattened complex manifolds and subsectors of $\mathcal{N}=4$ self-dual Yang-Mills theory

Christian Sämann
Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
E-mail: saemann@itp.uni-hannover.de

Abstract: In this paper, we propose so-called fattened complex manifolds as target spaces for the topological B-model. We naturally obtain these manifolds by restricting the structure sheaf of the $\mathcal{N}=4$ supertwistor space, a process, which can be understood as a fermionic dimensional reduction. Using the twistorial description of these fattened complex manifolds, we construct Penrose-Ward transforms between solutions to the holomorphic Chern-Simons equations on these spaces and bosonic subsectors of solutions to the $\mathcal{N}=4$ self-dual Yang-Mills equations on $\mathbb{C}^4$ or $\mathbb{R}^4$. Furthermore, we comment on Yau's theorem for these spaces.

Keywords: Chern-Simons Theories, Topological Strings, Superspaces, Integrable Field Theories.

Contents

1. Introduction

Calabi-Yau manifolds play an important rôle in topological string theory. Besides other interesting features as e.g. mirror symmetry, they are suited as target spaces for the so-called topological B-model. This B-model has been shown to be equivalent to holomorphic Chern-Simons (hCS) theory defined on its target space [1].

In [2], Witten studied the supermanifold $\mathbb{C}P^{3|4}$ which is a Calabi-Yau supermanifold and simultaneously a supertwistor space. For this space, the equivalence of the topological B-model with hCS theory still holds and the moduli space of classical solutions of hCS theory can be bijectively mapped to the moduli space of solutions to the $\mathcal{N} = 4$ supersymmetrically extended self-dual Yang-Mills (SDYM) equations via a Penrose-Ward correspondence[1] [2] (for the discussion beyond linearized level, see [4]). A crucially new

[1]For reviews of twistor theory and the Penrose-Ward correspondence, see [3].

aspect of this supertwistor description is the possibility of giving an action for hCS theory on the Calabi-Yau supermanifold, which is due to the existence of a holomorphic volume form on that space. This has never been achieved on the purely bosonic twistor space.

The link between topological string theory and SDYM theory was subsequently used to calculate gauge theory amplitudes with string theory machinery and other methods [5]–[11]; even gravity amplitudes have been considered [12]. A variety of further issues appearing in the context of twistor strings has also been examined, see [13] or [14] for recent examples.

With the extensive use of Calabi-Yau supermanifolds, the question of how mirror symmetry fits into the picture became more and more interesting [15, 16]. Particularly for the discussion of this point, it is useful to have more spaces at hand which bring along the helpful feature of a twistorial description.

Besides the Calabi-Yau supermanifold $\mathbb{C}P^{3|4}$ introduced in [2] and the weighted projective superspaces proposed already in [2] and considered in [17, 16] and [18], there is another class of presumably interesting spaces in direct reach: the so-called "fattened complex manifolds" [19]. These manifolds are extensions of ordinary manifolds with additional dimensions described by even nilpotent coordinates. Similar spaces have been studied in the mathematical literature since the early 1960s [20]. A class of examples of them can formally be obtained by pairing the Graßmann coordinates of a supermanifold. But fattened complex manifolds even allow for more constructive freedom: One can define a twistorial Calabi-Yau supermanifold corresponding to a fattened complex manifold with additional fermionic extensions. On the fattened complex manifolds which we will study in the following, hCS theory corresponds to bosonic subsectors of $\mathcal{N} = 4$ SDYM theory similarly to the case of weighted projective spaces. However, the parity of the truncated field content and thus also the equations of motion will differ from some of the results obtained in [17] for weighted projective spaces.

In this paper, we first discuss the existing framework for exotic supermanifolds, i.e. generalized supermanifolds which have additional even nilpotent coordinates. We then briefly review supertwistor geometry before constructing hCS and SDYM theory for two cases of exotic supermanifolds via a twistor correspondence. In the real case, we give a field expansion for the gauge potential of hCS theory, making the relation to SDYM theory explicit. We close with some remarks on the extension of Yau's theorem to exotic supermanifolds which are Calabi-Yau. This theorem guarantees the existence of a Ricci-flat metric in every Kähler class for Kähler manifolds with vanishing first Chern class. The results we find are very similar to [21]: The straightforward extension of Yau's theorem to exotic supermanifolds is not valid in general, but an additional constraint has to be imposed.

2. Exotic supermanifolds

In this section, we want to give a brief review of the existing extensions or generalizations of supermanifolds, which are suited as a general framework for describing the target spaces for the B-model employed later on. In the following, we call every (in a well-defined way generalized) manifold which is locally described by k even, l even and nilpotent and q odd

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and nilpotent coordinates an *exotic supermanifold* of dimension $(k \oplus l|q)$. For a review on Graßmann algebras, supernumbers and supermanifolds, see e.g. [22].

2.1 Partially formal supermanifolds

The objects of supermathematics, as e.g. supermanifolds or supergroups, are naturally described as covariant functors from the category of Graßmann algebras to corresponding categories of ordinary mathematical objects, as manifolds or groups, [23]. A generalization of this setting is to consider covariant functors with the category of *almost nilpotent* (AN) *algebras*[2] as domain [24]. Recall that an AN algebra Ξ can be decomposed into an even part Ξ_0 and an odd part Ξ_1 as well as in the canonically embedded ground field (i.e. $\mathbb{R}$ or $\mathbb{C}$), Ξ_B, and the nilpotent part Ξ_S. The parts of elements $\xi \in \Xi$ belonging to Ξ_B and Ξ_S are called the *body* and the *soul* of ξ, respectively.

A *superspace* is a covariant functor from the category of AN algebras to the category of sets. Furthermore, a *topological superspace* is a functor from the category of AN algebras to the category of topological spaces.

Consider now a tuple $(x^1, \ldots, x^k, y^1, \ldots, y^l, \zeta^1, \ldots, \zeta^q)$ of k even, l even and nilpotent and q odd and nilpotent elements of an AN algebra Ξ, i.e. $x^i \in \Xi_0$, $y^i \in \Xi_0 \cap \Xi_S$ and $\zeta^i \in \Xi_1$. The functor from the category of AN algebras to such tuples is a superspace denoted by $\mathbb{R}^{k \oplus l|q}$. An open subset $U^{k \oplus l|q}$ of $\mathbb{R}^{k \oplus l|q}$, which is obtained by restricting the fixed ground field Ξ_B of the category of AN algebras to an open subset, is called a *superdomain* of dimension $(k \oplus l|q)$. After defining a graded basis $(e_1, \ldots, e_k, f_1, \ldots, f_l, \varepsilon_1, \ldots, \varepsilon_q)$ consisting of $k + l$ even and q odd vectors, one can consider the set of linear combinations $\left\{x^i e_i + y^j f_j + \zeta^\alpha \varepsilon_\alpha\right\}$ which forms a *supervector space* [24].

Roughly speaking, one defines a partially formal supermanifold[3] of dimensions $(k \oplus l|q)$ as a topological superspace smoothly glued together from superdomains $U^{k \oplus l|q}$. Although we will not need the exact definition in the subsequent discussion, we will nevertheless give it here for completeness sake.

We define a *map between two superspaces* as a natural transformation of functors. More explicitly, consider two superspaces $\mathcal{M}$ and $\mathcal{N}$. Then a map $F : \mathcal{M} \to \mathcal{N}$ is a map between superspaces, if F is compatible with the morphisms of AN algebras $\alpha : \Xi \to \Xi'$. We call a smooth map $\kappa : \mathbb{R}^{k \oplus l|q}_\Xi \to \mathbb{R}^{k' \oplus l'|q'}_\Xi$ between two superdomains Ξ_0-*smooth*, if for every $x \in \mathbb{R}^{k \oplus l|q}_\Xi$ the tangent map $(\kappa_\Xi)_* : T_x \to T_{\kappa_\Xi(x)}$ is a homomorphism of Ξ_0-modules. Furthermore, we call a map $\kappa : \mathbb{R}^{k \oplus l|q} \to \mathbb{R}^{k' \oplus l'|q'}$ *smooth*, if for all AN algebras Ξ the maps κ_Ξ are Ξ_0-smooth.

Now we can be more precise: A *partially formal supermanifold of dimension* $(k \oplus l|q)$ is a superspace locally equivalent to superdomains of dimension $(k \oplus l|q)$ with smooth transition functions on the overlaps. Thus, a partially formal supermanifold is also an exotic supermanifold.

[2] An *almost nilpotent algebra* is an associative, finite-dimensional, unital, $\mathbb{Z}_2$-graded supercommutative algebra in which the ideal of nilpotent elements has codimension 1.

[3] This term was introduced in [25].

However, not every exotic supermanifold is partially formal. We will shortly encounter examples of such cases: exotic supermanifolds, which are constructed using a particular AN algebra instead of working with the category of AN algebras.

The definitions used in this section stem from [24], where one also finds examples of applications.

Unfortunately, it is not clear how to define a general integration over the even nilpotent part of such spaces; even the existence of such an integral is questionable. We will comment on this point later on. As we need an integration to define an action for our models, we have to turn to other generalizations.

2.2 Thick complex manifolds

Extensions to m-th formal neighborhoods of a submanifold X in a manifold $Y \supset X$ and the more general thickening procedure have been proposed and considered long ago[4] in the context of twistor theory, in particular for ambitwistor spaces, e.g. in [28]–[31]. We will ignore this motivation and only recollect the definitions needed for our subsequent discussion.

Given a complex manifold X with structure sheaf $\mathcal{O}_X$, we consider a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_{(m)}$ on X with a homomorphism $\alpha : \mathcal{O}_{(m)} \to \mathcal{O}_X$, such that locally $\mathcal{O}_{(m)}$ is isomorphic to $\mathcal{O}[y]/(y^{m+1})$ where y is a formal (complex) variable and α is the obvious projection. The resulting ringed space $X_{(m)} := (X, \mathcal{O}_{(m)})$ is called a *thick complex manifold*. Similarly to the nomenclature of supermanifolds, we call the complex manifold X the *body* of $X_{(m)}$.

As a simple example, let X be a closed submanifold of the complex manifold Y with codimension one. Let $\mathcal{I}$ be the ideal of functions vanishing on X. Then $\mathcal{O}_{(m)} = \mathcal{O}_Y/\mathcal{I}^{m+1}$ is called an *infinitesimal neighborhood* or the m-th *formal neighborhood* of X. This is a special case of a thick complex manifold. Assuming that X has complex dimension n, $\mathcal{O}_{(m)}$ is also an exotic supermanifold of dimension $(n \oplus 1|0)$. More explicitly, let $(x^1, \ldots, x^n)$ be local coordinates on X and $(x^1, \ldots, x^n, y)$ local coordinates on Y. Then the ideal $\mathcal{I}$ is generated by y and $\mathcal{O}_{(m)}$ is locally a formal polynomial in y with coefficients in $\mathcal{O}_X$ together with the identification $y^{m+1} \sim 0$. Furthermore, one has $\mathcal{O}_{(0)} = \mathcal{O}_X$.

Returning to the local description as a formal polynomial in y, we note that there is no object y^{-1} as it would violate associativity by an argument like $0 = y^{-1}y^{m+1} = y^{-1}yy^m = y^m$. However, the inverse of a formal polynomial in y is defined if (and only if) the zeroth order monomial has an inverse. Suppose $p = a + \sum_{i=1}^m f_i y^i = a + b$, then we have $p^{-1} = \frac{1}{a}\sum_{i=0}^m (-\frac{b}{a})^i$, analogously to the inverse of a supernumber.

A *holomorphic vector bundle* on $(X, \mathcal{O}_{(m)})$ is a locally free sheaf of $\mathcal{O}_{(m)}$-modules.

The *tangent space* of a thick complex manifold is the sheaf of derivations $D : \mathcal{O}_{(m)} \to \mathcal{O}_{(m)}$. Let us consider again our above example $X_{(m)} = (X, \mathcal{O}_{(m)})$. Locally, an element of $TX_{(m)}$ will take the form $D = f\frac{\partial}{\partial y} + \sum_j g^j \frac{\partial}{\partial x^j}$ together with the differentiation rules

$$\frac{\partial}{\partial y} y = 1\,, \qquad \frac{\partial}{\partial y} x^i = \frac{\partial}{\partial x^i} y = 0\,, \qquad \frac{\partial}{\partial x^i} x^j = \delta_i^j\,. \tag{2.1}$$

[4] In fact, the study of infinitesimal neighborhoods goes back to [20] and [26]. For a recent review, see [27].

All this and the introduction of cotangent spaces for thick complex manifolds is found in [31].

In defining a (definite) integral over the nilpotent formal variable y, which is needed for formulating hCS theory by giving an action, one faces the same difficulty as in the case of Berezin integration: the integral should not be taken over a specific range as we integrate over an infinitesimal neighborhood which would give rise to infinitesimal intervals. Furthermore, this neighborhood is purely formal and so has to be the integration. Recall that a suitable integration I should satisfy the rule[5] $DI = ID = 0$, where D is a derivative with respect to a variable over which I integrates. The first requirement $DI = 0$ states that the result of definite integration does not depend on the variables integrated over. The requirement $ID = 0$ for integration domains with vanishing boundary (or functions vanishing on the boundary) is the foundation of Stokes' formula and integration by parts. It is easy to see that the condition $DI = ID = 0$ demands that

$$I = c \cdot \frac{\partial^m}{\partial y^m}\,, \tag{2.2}$$

where y is the local formal variable from the definition of $X_{(m)}$ and c is an arbitrary normalization constant, e.g. $c = 1/m!$ would be most convenient. Thus, we define

$$\int \mathrm{d}y f := \frac{1}{m!}\frac{\partial^m}{\partial y^m} f\,. \tag{2.3}$$

This definition only relies on an already well-defined operation and thus is well-defined itself.[6] Additionally, it also agrees with the intuitive picture, that the integral of a constant over an infinitesimal neighborhood should vanish. Integration over a thick complex manifold is an integro-differential operation.

Consider now a change of coordinates $(x^1, \ldots, x^n, y) \to (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y})$ which leaves invariant the structure of the thick complex manifold. That is, $\tilde{x}^i$ is independent of y, and $\tilde{y}$ is a polynomial only in y with vanishing zeroth order coefficient and non-vanishing first order coefficient. Because of $\partial_{\tilde{y}} = \frac{\partial y}{\partial \tilde{y}}\,\partial_y$, we have the following transformation of a volume element under such a coordinate change:

$$\mathrm{d}\tilde{x}^1 \cdots \mathrm{d}\tilde{x}^n \mathrm{d}\tilde{y} = \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \mathrm{d}x^1 \cdots \mathrm{d}x^n \left(\frac{\partial y}{\partial \tilde{y}}\right)^m \mathrm{d}y\,. \tag{2.4}$$

The theorems in [31] concerning obstructions to finding $X_{(m+1)}$ given $X_{(m)}$ will not be needed in the following, as we will mainly work with order one thickenings (or fattenings) and in the remaining cases, the existence directly follows by construction.

[5] This rule can also be used to fix Berezin integration.

[6] From this definition, we see the problem arising for partially formal supermanifolds: The integration process on thick complex manifolds returns the coefficient of the monomial with highest possible power in y. For partially formal supermanifolds, where one works with the category of AN algebras, such a highest power does not exist as it is different for each individual AN algebra.

2.3 Fattened complex manifolds

Fattened complex manifolds [19] are straightforward generalizations of thick complex manifolds. Consider again a complex manifold X with structure sheaf $\mathcal{O}_X$. The *m-th order fattening with codimension k* of X is the ringed space $X_{(m,k)} = (X, \mathcal{O}_{(m,k)})$ where $\mathcal{O}_{(m,k)}$ is locally isomorphic to $\mathcal{O}[y^1, \ldots, y^k]/(y^1, \ldots, y^k)^{m+1}$. Here the y^i are again formal complex variables. We also demand the existence of the (obvious) homomorphism $\alpha : \mathcal{O}_{(m,k)} \to \mathcal{O}_X$. It follows immediately, that a fattening with codimension 1 is a thickening. Furthermore, an (m,k)-fattening of an n-dimensional complex manifold X is an exotic supermanifold of dimension $(n \oplus k|0)$ and we call X the *body* of $X_{(m,k)}$.

As in the case of thick complex manifolds, there are no inverses for the y^i, but the inverse of a formal polynomial p in the y^i decomposed into $p = a+b$, where b is the nilpotent part of p, exists again if and only if $a \neq 0$ and it is then given by $p^{-1} = \frac{1}{a}\sum_{i=0}^{m}(-\frac{b}{a})^i$. A *holomorphic vector bundle* on $\mathcal{O}_{(m,k)}$ is a locally free sheaf of $\mathcal{O}_{(m,k)}$-modules. The *tangent space* of a thick complex manifold is also generalized in an obvious manner.

We define the integral analogously to thick complex manifolds as

$$\int \mathrm{d}y^1 \cdots \mathrm{d}y^k f := \frac{1}{m!}\frac{\partial^m}{\partial(y^1)^m}\cdots\frac{1}{m!}\frac{\partial^m}{\partial(y^k)^m} f \,. \tag{2.5}$$

A change of coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^k) \to (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}^1, \ldots, \tilde{y}^k)$ must again preserve the structure of the fat complex manifold: $\tilde{x}^i$ is independent of the y^i and the $\tilde{y}^i$ are nilpotent polynomials in the y^i with vanishing monomial of order 0 and at least one nonvanishing monomial of order 1. Evidently, all the $\tilde{y}^i$ have to be linearly independent. Such a coordinate transformation results in a more complicated transformation law for the volume element:

$$\mathrm{d}\tilde{x}^1 \cdots \mathrm{d}\tilde{x}^n \mathrm{d}\tilde{y}^1 \cdots \tilde{y}^k = \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \mathrm{d}x^1 \cdots \mathrm{d}x^n \left(\frac{\partial y^{i_1}}{\partial \tilde{y}^1} \cdots \frac{\partial y^{i_k}}{\partial \tilde{y}^k}\right)^m \mathrm{d}y^{i_1} \cdots \mathrm{d}y^{i_k} \,, \tag{2.6}$$

where a sum over the indices $(i_1, \ldots, i_k)$ is implied. In this case, the coefficient for the transformation of the nilpotent formal variables cannot be simplified. Recall that in the case of ordinary differential forms, the wedge product provides the antisymmetry needed to form the determinant of the Jacobi matrix. In the case of Berezin integration, the anticommutativity of the derivatives with respect to Graßmann variables does the same for the inverse of the Jacobi matrix. Here, we have neither of these and therefore no determinant appears.

2.4 Thick and fattened supermanifolds

After thickening or fattening a complex manifold, one can readily add fermionic dimensions. Given a thickening of an n-dimensional complex manifold of order m, the simplest example is possibly $\Pi T X_{(m)}$, an $(n \oplus 1|n+1)$ dimensional exotic supermanifold. However, we will not study such objects in the following.

2.5 Exotic Calabi-Yau supermanifolds

It has become common usage to call a supermanifold with vanishing first Chern class a Calabi-Yau supermanifold, even if not all such spaces admit a Ricci-flat metric. Counterexamples to Yau's theorem for Calabi-Yau supermanifolds can be found in [21]. Following this convention, we call an exotic supermanifold Calabi-Yau, if its first Chern class vanishes and it therefore comes with a holomorphic volume form. Also for exotic supermanifolds, the Calabi-Yau property is not sufficient for the existence of a Ricci-flat metric, as we will find in the last section.

Nevertheless, one should remark that vanishing of the first Chern class — and not Ricci-flatness — is necessary for a consistent definition of the B-model on a manifold. And, from another viewpoint, it is only with the help of a holomorphic volume form, that one can give an action for hCS theory. Thus, the nomenclature is justified from a physicist's point of view.

2.6 Dolbeault and Čech descriptions of holomorphic vector bundles

The twistor correspondence [3] makes heavy use of two different descriptions of holomorphic vector bundles: the Dolbeault and the Čech description. Let us briefly comment on the extension of both to fattened complex manifolds.

Consider a trivial principal G-bundle P over a fattened complex manifold X covered by a collection of patches $\mathfrak{U} = \{\mathcal{U}_a\}$ with coordinates (z_a^i, y_a^j) and let G have a representation in terms of $n \times n$ matrices.

Let $\mathfrak{G}$ be an arbitrary sheaf of G-valued functions on X. The set of *Čech q-cochains* $C^q(\mathfrak{U}, \mathfrak{G})$ is the collection $\psi = \{\psi_{a_0 \cdots a_q}\}$ of sections of $\mathfrak{G}$ defined on nonempty intersections $\mathcal{U}_{a_0} \cap \cdots \cap \mathcal{U}_{a_q}$. Furthermore, we define the sets of Čech 0- and 1-cocycles by[7]

$$\begin{aligned} Z^0(\mathfrak{U}, \mathfrak{G}) &:= \{\, \psi \in C^0(\mathfrak{U}, \mathfrak{G}) | \psi_a = \psi_b \qquad \text{on } \mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing \} = \Gamma(\mathfrak{U}, \mathfrak{G})\,, && (2.7)\\ Z^1(\mathfrak{U}, \mathfrak{S}) &:= \{\, \chi \in C^1(\mathfrak{U}, \mathfrak{G}) | \chi_{ab} = \chi_{ba}^{-1} \qquad \text{on } \mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing\,, \\ &\qquad\qquad \chi_{ab}\chi_{bc}\chi_{ca} = \mathbb{1} \qquad \text{on } \mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c \neq \varnothing \}\,. && (2.8) \end{aligned}$$

This definition implies, that the Čech 0-cocyles are independent of the covering: $Z^0(\mathfrak{U}, \mathfrak{G}) = Z^0(X, \mathfrak{G})$, and we define the *zeroth Čech cohomology set* by $H^0(X, \mathfrak{G}) := Z^0(X, \mathfrak{G})$. Two 1-cocycles $\tilde{\chi}$ and χ are called *equivalent* if there is a 0-cochain $\psi \in C^0(\mathfrak{U}, \mathfrak{G})$ such that $\tilde{\chi}_{ab} = \psi_a \chi_{ab} \psi_b^{-1}$ on all $\mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing$. Dividing $Z^1(\mathfrak{U}, \mathfrak{G})$ by this equivalence relation gives the *first Čech cohomology set* $H^1(\mathfrak{U}, \mathfrak{G}) \cong Z^1(\mathfrak{U}, \mathfrak{G})/C^0(\mathfrak{U}, \mathfrak{G})$.

Besides the Čech cohomology sets, we also need the sheaf $\mathfrak{S}$ of smooth G-valued functions on X and its subsheaf $\mathfrak{H}$ of holomorphic G-valued functions on X. Furthermore,

[7]Note that any regular matrix-valued function on a fattened complex manifold of order m can be decomposed into an ordinary matrix-valued function and a nilpotent matrix-valued function, both defined on the body of the fattened complex manifold: $\psi = \psi_0 + \psi_n$. The inverse of such a function ψ exists, if ψ_0 is invertible and then it is explicitly given by

$$\psi^{-1} = \psi_0^{-1} - \psi_0^{-1}\psi_n\psi_0^{-1} + \psi_0^{-1}\psi_n\psi_0^{-1}\psi_n\psi_0^{-1} - \psi_0^{-1}\psi_n\psi_0^{-1}\psi_n\psi_0^{-1}\psi_n\psi_0^{-1} + \cdots\,,$$

with $m+1$ terms altogether.

we denote by $\mathfrak{A}$ the sheaf of flat (0,1)-connections on P, i.e. germs of solutions to

$$\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0\,, \tag{2.9}$$

which are purely holomorphic in the fattened directions, i.e.

$$\frac{\partial}{\partial \bar{y}^j_a}\,\mathcal{A}^{0,1}_a = 0 \qquad \text{on all } \mathcal{U}_a\,. \tag{2.10}$$

Note that elements $\mathcal{A}^{0,1}$ of $H^0(X,\mathfrak{A}) = \Gamma(X,\mathfrak{A})$ define a holomorphic structure $\bar{\partial}_{\mathcal{A}} = \bar{\partial} + \mathcal{A}^{0,1}$ on a trivial rank n complex vector bundle over X. The moduli space $\mathcal{M}$ of such holomorphic structures is obtained by dividing $H^0(X,\mathfrak{A})$ by the group of gauge transformations, i.e. elements g of $H^0(X,\mathfrak{S})$ acting on elements $\mathcal{A}^{0,1}$ of $H^0(X,\mathfrak{A})$ as

$$\mathcal{A}^{0,1} \mapsto g\mathcal{A}^{0,1}g^{-1} + g\bar{\partial}g^{-1}\,. \tag{2.11}$$

Thus, we have $\mathcal{M} \cong H^0(X,\mathfrak{A})/H^0(X,\mathfrak{S})$ and this is the Dolbeault description of holomorphic vector bundles.

Contrary to the connections used in the Dolbeault description, the Čech description uses transition function to define vector bundles. Clearly, such a transition function has to belong to the first Čech cocyle set of a suitable sheaf $\mathfrak{G}$. Furthermore, we want to call two vector bundles equivalent, if there exists an element h of $C^0(\mathfrak{U},\mathfrak{G})$ such that

$$f_{ab} = h_a^{-1}\tilde{f}_{ab}h_b \qquad \text{on all } \mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing\,. \tag{2.12}$$

Thus, we observe that holomorphic and smooth vector bundles have transition functions which are elements of the Čech cohomology sets $H^1(\mathfrak{U},\mathfrak{H})$ and $H^1(\mathfrak{U},\mathfrak{S})$, respectively.

If the patches $\mathcal{U}_a$ of the covering $\mathfrak{U}$ are Stein manifolds, one can show that the first Čech cohomology sets are independent of the covering and depend only on the manifold X, e.g. $H^1(\mathfrak{U},\mathfrak{S}) = H^1(X,\mathfrak{S})$. This is well known to be the case for purely bosonic twistor spaces and since the covering is obviously unaffected by the extension to an infinitesimal neighborhood,[8] we refrain from going into too much detail at this point.

To connect both descriptions, let us first introduce the subset $\mathfrak{X}$ of $C^0(\mathfrak{U},\mathfrak{S})$ given by a collection of G-valued functions $\psi = \{\psi_a\}$, which fulfill

$$\psi_a\bar{\partial}\psi_a^{-1} = \psi_b\bar{\partial}\psi_b^{-1}\,. \tag{2.13}$$

Due to (2.9)–(2.10), elements of $H^0(X,\mathfrak{A})$ can be written as $\psi\bar{\partial}\psi^{-1}$ with $\psi \in \mathfrak{X}$. Thus, for every $\mathcal{A}^{0,1} \in H^0(X,\mathfrak{A})$ we have corresponding elements $\psi \in \mathfrak{X}$. Now, one of these ψ can be used to define the transition functions of a rank n holomorphic vector bundle $\mathcal{E}$ over X by the formula

$$f_{ab} = \psi_a^{-1}\psi_b \qquad \text{on } \mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing\,. \tag{2.14}$$

It is easily checked, that the f_{ab} constructed in this way are holomorphic. Furthermore, they define holomorphic vector bundles which are topologically trivial, but not holomorphically trivial. Thus, they belong to the kernel of a map $\rho : H^1(X,\mathfrak{H}) \to H^1(X,\mathfrak{S})$.

[8] An infinitesimal neighborhood cannot be covered partially.

The isomorphy between the moduli spaces of both descriptions is easily found. We have the short exact sequence

$$0 \to \mathfrak{H} \xrightarrow{i} \mathfrak{S} \xrightarrow{\delta^0} \mathfrak{A} \xrightarrow{\delta^1} 0\,, \tag{2.15}$$

where i denotes the embedding of $\mathfrak{H}$ in $\mathfrak{S}$, δ^0 is the map $\mathfrak{S} \ni \psi \mapsto \psi\bar{\partial}\psi^{-1} \in \mathfrak{A}$ and δ^1 is the map $\mathfrak{A} \ni \mathcal{A}^{0,1} \mapsto \bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1}$ (cf. [32] for the purely bosonic case). This short exact sequence induces a long exact sequence of cohomology groups

$$0 \to H^0(X,\mathfrak{H}) \xrightarrow{i_*} H^0(X,\mathfrak{S}) \xrightarrow{\delta^0_*} H^0(X,\mathfrak{A}) \xrightarrow{\delta^1_*} H^1(X,\mathfrak{H}) \xrightarrow{\rho} H^1(X,\mathfrak{S}) \to \cdots\,, \tag{2.16}$$

and from this we see that $\ker\rho \cong H^0(X,\mathfrak{A})/H^0(X,\mathfrak{S})$. Thus, the moduli spaces of both descriptions are isomorphic, i.e. the moduli space of holomorphic vector bundles smoothly equivalent to the trivial bundle is bijective to the moduli space of hCS theory on X.

3. Twistor correspondence for fattened complex manifolds

3.1 Twistor geometry

The purpose of this section is to fix our notation. For a more extensive review on twistors, supertwistors and the twistor correspondence in conventions very similar to the ones used here, see e.g. [4] and the references therein.

Consider the projective space $\mathbb{C}P^3$ described by homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha})$ with indices $\alpha, \dot\alpha = 1, 2$. Let $\mathcal{P}^3$ be the subspace of $\mathbb{C}P^3$ for which $(\lambda_{\dot\alpha}) \neq (0,0)^T$ and let it be covered by two patches U_+ and U_- for which $\lambda_{\dot 1} \neq 0$ and $\lambda_{\dot 2} \neq 0$, respectively. Then we can introduce inhomogeneous coordinates $(z^\alpha_+, \lambda_+) := (\omega^\alpha/\lambda_{\dot 1}, \lambda_{\dot 2}/\lambda_{\dot 1})$ on U_+ and $(z^\alpha_-, \lambda_-) := (\omega^\alpha/\lambda_{\dot 2}, \lambda_{\dot 1}/\lambda_{\dot 2})$ on U_-. Note that $z^\alpha_+ = \lambda_+ z^\alpha_-$ and $\lambda_+ = (\lambda_-)^{-1}$ on $U_+ \cap U_-$, and thus the space $\mathcal{P}^3$ is the rank two vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ over the Riemann sphere $\mathbb{C}P^1$. Holomorphic sections of this bundle are parametrized by elements $(x^{\alpha\dot\alpha})$ of the space $\mathbb{C}^4$ and locally defined by the equations

$$z^\alpha_+ = x^{\alpha\dot\alpha}\lambda^+_{\dot\alpha} \qquad \text{and} \qquad z^\alpha_- = x^{\alpha\dot\alpha}\lambda^-_{\dot\alpha}\,, \tag{3.1}$$

where we introduced the notation $(\lambda^+_{\dot\alpha}) = (1, \lambda_+)^T$ and $(\lambda^-_{\dot\alpha}) = (\lambda_-, 1)^T$. Using these equations, one can establish a double fibration

$$\begin{tikzcd} & \mathbb{C}^4 \times \mathbb{C}P^1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathcal{P}^3 & & \mathbb{C}^4 \end{tikzcd} \tag{3.2}$$

with obvious projections π_1 and π_2. The tangent bundle $T^{(0,1)}\mathcal{P}^3$ has local sections $\frac{\partial}{\partial \bar{z}^\alpha_\pm}$ and $\frac{\partial}{\partial \bar{\lambda}_\pm}$ and the (1,0)-part of the tangent space to the leaves of the projection π_2, i.e. $\ker \pi_{2*}$, is locally spanned by $V^\pm_\alpha = \lambda^{\dot\alpha}_\pm \frac{\partial}{\partial x^{\alpha\dot\alpha}}$, where we used $\lambda^{\dot\alpha}_\pm = \varepsilon^{\dot\alpha\dot\beta}\lambda^\pm_{\dot\beta}$ and $\varepsilon^{\dot 1\dot 2} = -1$.

To obtain the $\mathcal{N}$-extended supertwistor space [33], we replace the rank two vector bundle $\mathcal{P}^3 = \mathbb{C}^2 \otimes \mathcal{O}(1)$ by[9] $\mathcal{P}^{3|\mathcal{N}} := \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \Pi\left(\mathbb{C}^{\mathcal{N}} \otimes \mathcal{O}(1)\right)$. This space is again covered by two patches $\mathcal{U}_+$ and $\mathcal{U}_-$ with the same bosonic coordinates and sections as $\mathcal{P}^3$. Local fermionic coordinates are $\eta_k^\pm$ with $\eta_k^+ = \lambda_+ \eta_k^-$ on $\mathcal{U}_+ \cap \mathcal{U}_-$, where $k = 1, \ldots, \mathcal{N}$, and fermionic sections are parameterized by elements $(\eta_k^{\dot{\alpha}})$ of the space $\mathbb{C}^{0|2\mathcal{N}}$ as $\eta_k^\pm = \eta_k^{\dot{\alpha}} \lambda_{\dot{\alpha}}^\pm$. The total moduli space of sections of the vector bundle $\mathcal{P}^{3|\mathcal{N}} \to \mathbb{C}P^1$ is $\mathbb{C}^{4|2\mathcal{N}}$ and the double fibration (3.2) is extended to

$$\begin{tikzcd} & \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}^{4|2\mathcal{N}} \end{tikzcd} \tag{3.3}$$

Furthermore, we get the additional fermionic vector fields $\frac{\partial}{\partial \bar{\eta}_k^\pm}$ on $\mathcal{P}^{3|\mathcal{N}}$ and the $(1,0)$-vector fields $D_\pm^k := \lambda_\pm^{\dot{\alpha}} \frac{\partial}{\partial \eta_k^{\dot{\alpha}}}$ along the leaves of the projection π_2.

Instead of the shorthand notation $\mathcal{P}^{3|\mathcal{N}}$, we will often write $(\mathcal{P}^3, \mathcal{O}_{[\mathcal{N}]})$ in the following, which makes the extension of the structure sheaf of $\mathcal{P}^3$ explicit. The sheaf $\mathcal{O}_{[\mathcal{N}]}$ is locally the tensor product of the structure sheaf of a patch of the assigned space $\mathcal{P}^3$ (which is suppressed in our notation) and a Graßmann algebra of $\mathcal{N}$ generators.

Let us roughly outline a typical twistor correspondence. For this, consider again the double fibration (3.2). We start with the Dolbeault description of a trivial complex vector bundle E over $\mathcal{P}^3$, i.e. we consider solutions $A^{0,1}$ to the equations of motion $\bar{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0$ of hCS theory on $\mathcal{P}^3$. These solutions can be written as $A_\pm^{0,1} = \psi_\pm \bar{\partial} \psi_\pm^{-1}$ on each patch, and the functions $\psi_\pm$ can be used to construct a transition function $f_{+-} = \psi_+^{-1} \psi_-$ for a holomorphic vector bundle $\tilde{E}$, which is smoothly equivalent to E. Thus we switched from a bundle in the Dolbeault description to an equivalent bundle in the Čech description. In the following, we restrict ourselves to solutions $A^{0,1}$ for which the component $A_{\bar{\lambda}_\pm}^{0,1} := \frac{\partial}{\partial \bar{\lambda}_\pm} \lrcorner A^{0,1}$ vanishes. These correspond to bundles $\tilde{E}$, which are holomorphically trivial when restricted to any projective line $\mathbb{C}P_x^1 \hookrightarrow \mathcal{P}^3$, a demand which is essential in the subsequent construction. After pulling this bundle back to $\mathbb{C}^4 \times \mathbb{C}P^1$, we use the holomorphical triviality of $\tilde{E}|_{\mathbb{C}P_x^1}$ to choose a different gauge $\psi \to \hat{\psi}$ in which the $\hat{\psi}$s are purely holomorphic functions of the coordinates on $\mathbb{C}^4 \times \mathbb{C}P^1$. From these $\hat{\psi}$, we obtain again a (gauge transformed) Dolbeault description $\hat{A}_\alpha^\pm = \hat{\psi}_\pm V_\alpha^\pm \hat{\psi}_\pm^{-1}$ of the bundle E pulled back to $\mathbb{C}^4 \times \mathbb{C}P^1$, which can be readily pushed forward to $\mathbb{C}^4$. Then the gauge potential obtained in this way satisfies the self-duality equations on $\mathbb{C}^4$. Altogether, there is a one-to-one correspondence between the moduli space of solutions (with $A_{\bar{\lambda}_\pm}^{0,1} = 0$) to the hCS equations of motion on $\mathcal{P}^3$ on the one hand side and the moduli space of solutions to the SDYM equations on $\mathbb{C}^4$ on the other hand side, where the moduli spaces are obtained by dividing the solution spaces by the respective group of gauge transformations.

3.2 From the supertwistor space $\mathcal{P}^{3|4}$ to fattened complex manifolds

The Calabi-Yau property, i.e. vanishing of the first Chern class or equivalently the existence

[9] The operator Π inverts the parity of the fibre coordinates in a fibre bundle, see [22, 4].

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of a globally well-defined holomorphic volume form, is essential for defining the B-model on a certain space. Consider the space $\mathcal{P}^{3|4}$ as introduced in the last section. Since the volume element Ω which is locally given by $\Omega_\pm := \pm \mathrm{d}z^1_\pm \wedge \mathrm{d}z^2_\pm \wedge \mathrm{d}\lambda_\pm \wedge \mathrm{d}\eta^\pm_1 \wedge \cdots \wedge \mathrm{d}\eta^\pm_4$ is globally defined and holomorphic, $\mathcal{P}^{3|4}$ is a Calabi-Yau supermanifold.[10] Other spaces which have a twistorial $\mathcal{O}(1)\oplus\mathcal{O}(1)$ body and are still Calabi-Yau supermanifolds are, e.g., the weighted projective spaces[11] $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)$ with (p,q) equal to $(1,3)$, $(2,2)$ and $(4,0)$ as considered in [17]. The B-model on these manifolds was shown to be equivalent to $\mathcal{N}=4$ SDYM theory with a truncated field content. Additionally in the cases $(2,2)$ and $(4,0)$, the parity of some fields is changed, similarly to the result of a topological twist.

An obvious idea to obtain even more Calabi-Yau supermanifolds directly from $\mathcal{P}^{3|4}$ is to combine several fermionic variables into a single one,[12] e.g. to consider coordinates $(\zeta_1 := \eta_1, \zeta_2 := \eta_2\eta_3\eta_4)$. In an analogous situation for bosonic variables, one could always at least locally find additional coordinates complementing the reduced set to a set describing the full space. Fixing the complementing coordinates to certain values then means, that one considers a subvariety of the full space. However, as there is no inverse of Graßmann variables, the situation here is different. Instead of taking a subspace, we rather restrict the algebra of functions (and similarly the set of differential operators) by demanding a certain dependence on the Graßmann variables. One can indeed find complementing sets of functions to restore the full algebra of functions on $\mathcal{P}^{3|4}$. Underlining the argument that we do not consider a subspace of $\mathcal{P}^{3|4}$ is the observation that we still have to integrate over the full space $\mathcal{P}^{3|4}$: $\int \mathrm{d}\zeta_1\mathrm{d}\zeta_2 = \int \mathrm{d}\eta_1\cdots\mathrm{d}\eta_4$. This picture has a slight similarity to the definition of the body of a supermanifold as given in [35, 22].

Possible inequivalent groupings of the Graßmann coordinates of $\mathcal{P}^{3|4}$ are the previously given example $(\zeta_1 := \eta_1, \zeta_2 := \eta_2\eta_3\eta_4)$ as well as $(\zeta_1 = \eta_1, \zeta_2 = \eta_2, \zeta_3 = \eta_3\eta_4)$, $(\zeta_1 = \eta_1\eta_2, \zeta_2 = \eta_3\eta_4)$, and $(\zeta_1 = \eta_1\eta_2\eta_3\eta_4)$. They correspond to exotic supermanifolds of dimension $(3{\oplus}0|2)$, $(3{\oplus}1|2)$, $(3{\oplus}2|0)$, and $(3{\oplus}1|0)$, respectively. Considering hCS theory on them, one finds that the first one is equivalent to the case $W\mathbb{C}P^{3|2}(1,1,1,1|1,3)$ which was already discussed in [17]. The case $(3\oplus 2|0)$ will be similar to a the case $W\mathbb{C}P^{3|2}(1,1,1,1|2,2)$, but with a field content of partially different parity. The case $(3\oplus 1|2)$ is a mixture easily derived from combining the full case $\mathcal{P}^{3|4}$ with the case $(3\oplus 2|0)$. We restrict ourselves in the following to the cases $(3\oplus 2|0)$ and $(3\oplus 1|0)$.

Instead of considering independent twistor correspondences between fattened complex manifolds and the moduli space of relative deformations of the contained $\mathbb{C}P^1$, we will focus on *reductions* of the correspondence between $\mathcal{P}^{3|4}$ and $\mathbb{C}^{4|8}$. This formulation allows for a more direct identification of the remaining subsectors of $\mathcal{N}=4$ self-dual Yang-Mills theory and can in a sense be understood as a fermionic dimensional reduction.

[10] The total Chern class of $\mathcal{P}^{3|4}$ indeed adds up to zero: The body $\mathcal{O}(1)\oplus\mathcal{O}(1)\to\mathbb{C}P^1$ contributes 1 from each of the fibres of the line subbundles and 2 from the cotangent space of the sphere. The fermionic fibres contribute -4 altogether, as $\Pi\mathcal{O}(1)$ has Chern class -1 due to the inverse appearance of the equivalent of the Jacobi determinant in Berezin integration.

[11] In fact, one rather considers their open subspaces $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)\backslash W\mathbb{C}P^{1|2}(1,1|p,q)$.

[12] A similar situation has been considered in [34], where all the fermionic variables where combined into a single even nilpotent one.

3.3 The B-model on $\mathcal{P}^{3\oplus 2|0}$

3.3.1 Geometrical considerations

The starting point of our discussion is the supertwistor space $\mathcal{P}^{3|4} = (\mathcal{P}^3, \mathcal{O}_{[4]})$. Consider the differential operators

$$\mathcal{D}^{i1}_\pm := \eta_1^\pm \eta_2^\pm \frac{\partial}{\partial \eta_i^\pm} \qquad \text{and} \qquad \mathcal{D}^{i2}_\pm := \eta_3^\pm \eta_4^\pm \frac{\partial}{\partial \eta_{i+2}^\pm} \qquad \text{for } i = 1,2\,, \tag{3.4}$$

which are maps $\mathcal{O}_{[4]} \to \mathcal{O}_{[4]}$. The space $\mathcal{P}^3$ together with the structure sheaf

$$\mathcal{O}_{(1,2)} := \bigcap_{i,j=1,2} \ker \mathcal{D}^{ij}_+ = \bigcap_{i,j=1,2} \ker \mathcal{D}^{ij}_-\,, \tag{3.5}$$

which is a reduction of $\mathcal{O}_{[4]}$, is the fattened complex manifold $\mathcal{P}^{3\oplus 2|0}$, covered by two patches $\mathcal{U}_+$ and $\mathcal{U}_-$ and described by local coordinates $(z^\alpha_\pm,\, \lambda_\pm,\, y_1^\pm := \eta_1^\pm\eta_2^\pm,\, y_2^\pm := \eta_3^\pm\eta_4^\pm)$. The two even nilpotent coordinates $y_i^\pm$ are each sections of the line bundle $\mathcal{O}(2)$ with the identification $(y_i^\pm)^2 \sim 0$.

As pointed out before, the coordinates $y_i^\pm$ do not allow for a complementing set of coordinates, and therefore it is not possible to use Leibniz calculus in the transition from the η-coordinates on $(\mathcal{P}^3, \mathcal{O}_{[4]})$ to the y-coordinates on $(\mathcal{P}^3, \mathcal{O}_{(1,2)})$. Instead, from the observation that

$$\begin{aligned} \eta_2^\pm \frac{\partial}{\partial y_1^\pm} &= \left.\frac{\partial}{\partial \eta_1^\pm}\right|_{\mathcal{O}_{(1,2)}}\,, \qquad & \eta_1^\pm \frac{\partial}{\partial y_1^\pm} &= -\left.\frac{\partial}{\partial \eta_2^\pm}\right|_{\mathcal{O}_{(1,2)}}\,,\\ \eta_4^\pm \frac{\partial}{\partial y_2^\pm} &= \left.\frac{\partial}{\partial \eta_3^\pm}\right|_{\mathcal{O}_{(1,2)}}\,, \qquad & \eta_3^\pm \frac{\partial}{\partial y_2^\pm} &= -\left.\frac{\partial}{\partial \eta_4^\pm}\right|_{\mathcal{O}_{(1,2)}}\,, \end{aligned} \tag{3.6}$$

one directly obtains the following identities on $(\mathcal{P}^3, \mathcal{O}_{(1,2)})$:

$$\frac{\partial}{\partial y_1^\pm} = \frac{\partial}{\partial \eta_2^\pm}\frac{\partial}{\partial \eta_1^\pm} \qquad \text{and} \qquad \frac{\partial}{\partial y_2^\pm} = \frac{\partial}{\partial \eta_4^\pm}\frac{\partial}{\partial \eta_3^\pm}\,. \tag{3.7}$$

Equations (3.6) are easily derived by considering an arbitrary section f of $\mathcal{O}_{(1,2)}$:

$$f = a^0 + a^1 y_1 + a^2 y_2 + a^{12} y_1 y_2 = a^0 + a^1 \eta_1\eta_2 + a^2\eta_3\eta_4 + a^{12}\eta_1\eta_2\eta_3\eta_4\,, \tag{3.8}$$

where we suppressed the $\pm$ labels for convenience. Acting, e.g., by $\frac{\partial}{\partial \eta_1}$ on f, we see that this equals an action of $\eta_2 \frac{\partial}{\partial y_1}$. It is then also obvious, that we can make the formal identification (3.7) on $(\mathcal{P}^3, \mathcal{O}_{(1,2)})$. Still, a few more comments on (3.7) are in order. These differential operators clearly map $\mathcal{O}_{(1,2)} \to \mathcal{O}_{(1,2)}$ and fulfill

$$\frac{\partial}{\partial y_i^\pm}\, y_j^\pm = \delta^i_j\,. \tag{3.9}$$

Note, however, that they do not quite satisfy the Leibniz rule, e.g.:

$$1 = \frac{\partial}{\partial y_1^\pm}\, y_1^\pm = \frac{\partial}{\partial y_1^\pm}\,(\eta_1^\pm\eta_2^\pm) \neq \left(\frac{\partial}{\partial y_1^\pm}\,\eta_1^\pm\right)\eta_2^\pm + \eta_1^\pm \left(\frac{\partial}{\partial y_1^\pm}\,\eta_2^\pm\right) = 0\,. \tag{3.10}$$

This does not affect the fattened complex manifold $\mathcal{P}^{3\oplus 2|0}$ at all, but it imposes an obvious constraint on the formal manipulation of expressions involving the y-coordinates rewritten in terms of the η-coordinates.

For the cotangent space, we have the identification $\mathrm{d}y_1^\pm = \mathrm{d}\eta_2^\pm \mathrm{d}\eta_1^\pm$ and $\mathrm{d}y_2^\pm = \mathrm{d}\eta_4^\pm \mathrm{d}\eta_3^\pm$ and similarly to above, one has to take care in formal manipulations, as integration is equivalent to differentiation.

As discussed in section 3.1, holomorphic sections of the bundle $\mathcal{P}^{3|4} \to \mathbb{C}P^1$ are described by moduli which are elements of the space $\mathbb{C}^{4|8} = (\mathbb{C}^4, \mathcal{O}_{[8]})$. After the above reduction, holomorphic sections of the bundle $\mathcal{P}^{3\oplus 2|0} \to \mathbb{C}P^1$ are defined by the equations

$$z_\pm^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm \qquad \text{and} \qquad y_i^\pm = y_i^{\dot\alpha\dot\beta}\lambda_{\dot\alpha}^\pm\lambda_{\dot\beta}^\pm \,. \tag{3.11}$$

While the Graßmann algebra of the coordinates $\eta_k^\pm$ of $\mathcal{P}^{3|4}$ immediately imposed a Graßmann algebra on the moduli $\eta_k^{\dot\alpha} \in \mathbb{C}^{0|8}$, the situation here is more subtle. We have[13] $y_1^{(\dot\alpha\dot\beta)} = \eta_1^{(\dot\alpha}\eta_2^{\dot\beta)}$ and from this, we already note that $(y_1^{\dot 1\dot 2})^2 \neq 0$ but only $(y_1^{\dot 1\dot 2})^3 = 0$. Thus, the moduli space is a fattening of order 1 in $y_1^{\dot 1\dot 1}$ and $y_1^{\dot 2\dot 2}$, but a fattening of order 2 in $y_1^{\dot 1\dot 2}$ which analogously holds for $y_2^{\dot\alpha\dot\beta}$. Furthermore, we have the additional identities

$$y_i^{\dot 1\dot 2}y_i^{\dot 1\dot 2} = -\frac{1}{2}y_i^{\dot 1\dot 1}y_i^{\dot 2\dot 2} \qquad \text{and} \qquad y_i^{\dot 1\dot 2}y_i^{\dot 2\dot 2} = y_i^{\dot 1\dot 2}y_i^{\dot 1\dot 1} = 0\,. \tag{3.12}$$

Additional conditions which appear when working with fattened complex manifolds are not unusual and similar problems were encountered, e.g., in the discussion of fattened ambitwistor spaces in [29].

More formally, one can introduce the differential operators

$$\mathcal{D}^{1c} = (\eta_1^{\dot\alpha}\partial_{\dot\alpha}^1 - \eta_2^{\dot\alpha}\partial_{\dot\alpha}^2)\,, \qquad \mathcal{D}^{2c} = (\eta_3^{\dot\alpha}\partial_{\dot\alpha}^3 - \eta_4^{\dot\alpha}\partial_{\dot\alpha}^4)\,, \tag{3.13}$$

$$\mathcal{D}^{1s} = (\partial_{\dot 1}^2\partial_{\dot 2}^1 - \partial_{\dot 2}^2\partial_{\dot 1}^1)\,, \qquad \mathcal{D}^{2s} = (\partial_{\dot 1}^4\partial_{\dot 2}^3 - \partial_{\dot 2}^4\partial_{\dot 1}^3) \tag{3.14}$$

which map $\mathcal{O}_{[8]} \to \mathcal{O}_{[8]}$, and consider the overlap of kernels

$$\mathcal{O}_{(1;2,6)} := \bigcap_{i=1,2} \big(\ker(D^{ic}) \cap \ker(D^{is})\big)\,. \tag{3.15}$$

The space $\mathbb{C}^4$ together with the structure sheaf $\mathcal{O}_{(1;2,6)}$, which is a reduction of $\mathcal{O}_{[8]}$, is exactly the moduli space described above, i.e. a fattened complex manifold $\mathbb{C}^{4\oplus 6|0}$ on which the coordinates $y_i^{\dot\alpha\dot\beta}$ satisfy the additional constrains (3.12).

Altogether, we have the following reduction of the full double fibration (3.3) for $\mathcal{N} = 4$:

$$\begin{tikzcd}[column sep=tiny]
& (\mathbb{C}^4\times\mathbb{C}P^1, \mathcal{O}_{[8]}\otimes\mathcal{O}_{\mathbb{C}P^1}) \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & & \longrightarrow & & (\mathbb{C}^4\times\mathbb{C}P^1, \mathcal{O}_{(1;2,6)}\otimes\mathcal{O}_{\mathbb{C}P^1}) \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
(\mathcal{P}^3, \mathcal{O}_{[4]}) & & (\mathbb{C}^4, \mathcal{O}_{[8]}) & & (\mathcal{P}^3, \mathcal{O}_{(1,2)}) & & (\mathbb{C}^4, \mathcal{O}_{(1;2,6)})
\end{tikzcd} \tag{3.16}$$

[13]The brackets $(\cdot)$ and $[\cdot]$ denote symmetrization and antisymmetrization, respectively, of the enclosed indices with appropriate weight.

where $\mathcal{O}_{\mathbb{C}P^1}$ is the structure sheaf of the Riemann sphere $\mathbb{C}P^1$. The tangent spaces along the leaves of the projection π_2 are spanned by the vector fields

$$\begin{aligned} V^\pm_\alpha &= \lambda^{\dot\alpha}_\pm \partial_{\alpha\dot\alpha}\,, \qquad & V^\pm_\alpha &= \lambda^{\dot\alpha}_\pm \partial_{\alpha\dot\alpha}\,, \\ D^k_\pm &= \lambda^{\dot\alpha}_\pm \frac{\partial}{\partial\eta^{\dot\alpha}_k}\,, \qquad & D^i_{\dot\beta\pm} &= \lambda^{\dot\alpha}_\pm \frac{\partial}{\partial y_i^{(\dot\alpha\dot\beta)}} \end{aligned} \tag{3.17}$$

in the left and right double fibration in (3.16), where $k = 1,\ldots,4$. Note that similarly to (3.6), we have the identities

$$\begin{aligned} \eta^{\dot\alpha}_2 \frac{\partial}{\partial y_1^{(\dot\alpha\dot\beta)}} &= \left.\frac{\partial}{\partial\eta^{\dot\beta}_1}\right|_{\mathcal{O}_{(1;2,6)}}\,, \qquad & \eta^{\dot\alpha}_1 \frac{\partial}{\partial y_1^{(\dot\alpha\dot\beta)}} &= -\left.\frac{\partial}{\partial\eta^{\dot\beta}_2}\right|_{\mathcal{O}_{(1;2,6)}}\,, \\ \eta^{\dot\alpha}_4 \frac{\partial}{\partial y_2^{(\dot\alpha\dot\beta)}} &= \left.\frac{\partial}{\partial\eta^{\dot\beta}_3}\right|_{\mathcal{O}_{(1;2,6)}}\,, \qquad & \eta^{\dot\alpha}_3 \frac{\partial}{\partial y_2^{(\dot\alpha\dot\beta)}} &= -\left.\frac{\partial}{\partial\eta^{\dot\beta}_4}\right|_{\mathcal{O}_{(1;2,6)}}\,, \end{aligned} \tag{3.18}$$

and it follows, e.g., that

$$D^1_\pm\big|_{\mathcal{O}_{(1;2,6)}} = \eta^{\dot\alpha}_2 D^1_{\dot\alpha\pm} \qquad \text{and} \qquad D^2_\pm\big|_{\mathcal{O}_{(1;2,6)}} = -\eta^{\dot\alpha}_1 D^1_{\dot\alpha\pm}\,. \tag{3.19}$$

3.3.2 Field theoretical considerations

The topological B-model on $\mathcal{P}^{3\oplus 2|0} = (\mathcal{P}^3, \mathcal{O}_{(1,2)})$ is equivalent to hCS theory on $\mathcal{P}^{3\oplus 2|0}$ since a reduction of the structure sheaf does not affect the arguments used for this equivalence in [1, 2]. Consider a trivial rank n complex vector bundle[14] $\mathcal{E}$ over $\mathcal{P}^{3\oplus 2|0}$ with a connection $\mathcal{A}$. The action for hCS theory on this space reads

$$S = \int_{\mathcal{P}^{3\oplus 2|0}_{\mathrm{ch}}} \Omega^{3\oplus 2|0} \wedge \left(\mathcal{A}^{0,1}\wedge\bar\partial\mathcal{A}^{0,1} + \frac{2}{3}\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\right), \tag{3.20}$$

where $\mathcal{P}^{3\oplus 2|0}_{\mathrm{ch}}$ is the subspace[15] of $\mathcal{P}^{3\oplus 2|0}$ for which $\bar y_i^\pm = 0$, $\mathcal{A}^{0,1}$ is the (0,1)-part of $\mathcal{A}$ and $\Omega^{3\oplus 2|0}$ is the holomorphic volume form, e.g. $\Omega^{3\oplus 2|0}_+ = \mathrm{d}z^1_+\wedge\mathrm{d}z^2_+\wedge\mathrm{d}\lambda_+\wedge\mathrm{d}y_1^+\wedge\mathrm{d}y_2^+$. The equations of motion read $\bar\partial\mathcal{A}^{0,1} + \mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1} = 0$ and solutions define a holomorphic structure $\bar\partial_{\mathcal{A}}$ on $\mathcal{E}$. Given such a solution $\mathcal{A}^{0,1}$, one can locally write $\mathcal{A}^{0,1}|_{\mathcal{U}_\pm} = \psi_\pm\bar\partial\psi_\pm^{-1}$ with regular matrix-valued functions $\psi_\pm$ smooth on the patches $\mathcal{U}_\pm$ and from the gluing condition $\psi_+\bar\partial\psi_+^{-1} = \psi_-\bar\partial\psi_-^{-1}$ on the overlap $\mathcal{U}_+\cap\mathcal{U}_-$, one obtains $\bar\partial\left(\psi_+^{-1}\psi_-\right) = 0$. Thus, $f_{+-} := \psi_+^{-1}\psi_-$ defines a transition function for a holomorphic vector bundle $\tilde{\mathcal{E}}$, which is (smoothly) equivalent to $\mathcal{E}$.

Consider now the pull-back of the bundle $\tilde{\mathcal{E}}$ along π_2 in (3.16) to the space $\mathbb{C}^{4|8}\times\mathbb{C}P^1$, i.e. the holomorphic vector bundle $\pi_2^*\tilde{\mathcal{E}}$ with transition function $\pi_2^* f_{+-}$ satisfying $V^\pm_\alpha(\pi_2^* f_{+-}) = D^k_\pm(\pi_2^* f_{+-}) = 0$. Let us suppose that the vector bundle $\pi_2^*\tilde{\mathcal{E}}$ becomes

[14] Note that the components of sections of ordinary vector bundles over a supermanifold are superfunctions. The same holds for the components of connections and transition functions.

[15] This restriction to a subspace holomorphic in the fermionic coordinates, i.e. a chiral subspace, was proposed in [2] and is related to self-duality.

holomorphically trivial[16] when restricted to sections $\mathbb{C}P^1_{x,\eta}\hookrightarrow\mathcal{P}^{3|4}$. This implies, that there is a splitting $\pi_2^* f_{+-} = \hat{\psi}_+^{-1}\hat{\psi}_-$, where $\hat{\psi}_\pm$ are group-valued functions which are holomorphic in the moduli $(x^{\alpha\dot{\alpha}}, \eta_k^{\dot{\alpha}})$ and $\lambda_\pm$. From the condition $V_\alpha^\pm(\pi_2^* f_{+-}) = D_\pm^k(\pi_2^* f_{+-}) = 0$ we obtain, e.g. on $\mathcal{U}_+$

$$
\begin{aligned}
\hat{\psi}_+ V_\alpha^+ \hat{\psi}_+^{-1} = \hat{\psi}_- V_\alpha^+ \hat{\psi}_-^{-1} &=: \lambda_+^{\dot{\alpha}} \hat{\mathcal{A}}_{\alpha\dot{\alpha}} =: \hat{\mathcal{A}}_\alpha^+ \,, \\
\hat{\psi}_+ D_+^k \hat{\psi}_+^{-1} = \hat{\psi}_- D_+^k \hat{\psi}_-^{-1} &=: \lambda_+^{\dot{\alpha}} \hat{\mathcal{A}}_{\dot{\alpha}}^k =: \hat{\mathcal{A}}_+^k \,, \\
\hat{\psi}_+ \partial_{\bar{\lambda}_+} \hat{\psi}_+^{-1} = \hat{\psi}_- \partial_{\bar{\lambda}_+} \hat{\psi}_-^{-1} &=: \hat{\mathcal{A}}_{\bar{\lambda}_+} = 0 \,, \\
\hat{\psi}_+ \partial_{\bar{x}^{\alpha\dot{\alpha}}} \hat{\psi}_+^{-1} = \hat{\psi}_- \partial_{\bar{x}^{\alpha\dot{\alpha}}} \hat{\psi}_-^{-1} &= 0 \,.
\end{aligned} \tag{3.21}
$$

Considering the reduced structure sheaves, we can rewrite the second line of (3.21), e.g. for $k=1$ as

$$
\eta_2^{\dot{\beta}} \hat{\psi}_+ D^1_{\dot{\beta}+} \hat{\psi}_+^{-1} = \eta_2^{\dot{\beta}} \hat{\psi}_- D^1_{\dot{\beta}+} \hat{\psi}_-^{-1} =: \eta_2^{\dot{\beta}} \lambda_+^{\dot{\alpha}} \hat{\mathcal{A}}^1_{\dot{\alpha}\dot{\beta}} \,, \tag{3.22}
$$

which yields $\eta_2^{\dot{\beta}} \hat{\mathcal{A}}^1_{\dot{\alpha}\dot{\beta}} = \hat{\mathcal{A}}^1_{\dot{\alpha}}$. From this equation (and similar ones for other values of k) and the well-known superfield expansion of $\hat{\mathcal{A}}^k_{\dot{\alpha}}$ (see e.g. [36]), one can now construct the superfield expansion of $\hat{\mathcal{A}}^i_{\dot{\alpha}\dot{\beta}}$ by dropping all the terms, which are not in the kernel of the differential operators $\mathcal{D}^{jc}$ and $\mathcal{D}^{js}$. This will give rise to a bosonic subsector of $\mathcal{N}=4$ SDYM theory.

To be more explicit, we can also use (3.22) and introduce the covariant derivative $\nabla_{\alpha\dot{\alpha}} := \partial_{\alpha\dot{\alpha}} + [\hat{\mathcal{A}}_{\alpha\dot{\alpha}}, \cdot]$ and the first order differential operator $\nabla^i_{\dot{\alpha}\dot{\beta}} := \partial^i_{\dot{\alpha}\dot{\beta}} + [\hat{\mathcal{A}}^i_{\dot{\alpha}\dot{\beta}}, \cdot]$, which allow us to rewrite the compatibility conditions of the linear system behind (3.21), (3.22) for the reduced structure sheaf as

$$
\begin{gathered}
[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] + [\nabla_{\alpha\dot{\beta}}, \nabla_{\beta\dot{\alpha}}] = 0 \,, \qquad \eta_m^{\dot{\gamma}} \left([\nabla^i_{\dot{\alpha}\dot{\gamma}}, \nabla_{\beta\dot{\beta}}] + [\nabla^i_{\dot{\beta}\dot{\gamma}}, \nabla_{\beta\dot{\alpha}}] \right) = 0 \,, \\
\eta_m^{\dot{\gamma}} \eta_n^{\dot{\delta}} \left([\nabla^i_{\dot{\alpha}\dot{\gamma}}, \nabla^j_{\dot{\beta}\dot{\delta}}] + [\nabla^i_{\dot{\beta}\dot{\gamma}}, \nabla^j_{\dot{\alpha}\dot{\delta}}] \right) = 0 \,,
\end{gathered} \tag{3.23}
$$

where $m = 2i-1, 2i$ and $n = 2j-1, 2j$. Note that $\nabla^i_{\dot{\alpha}\dot{\beta}}$ is no true covariant derivative, as $\partial^i_{\dot{\alpha}\dot{\beta}}$ and $\hat{\mathcal{A}}^i_{\dot{\alpha}\dot{\beta}}$ do not have the same symmetry properties in the indices. Nevertheless, the differential operators $\nabla_{\alpha\dot{\alpha}}$ and $\nabla^i_{\dot{\alpha}\dot{\beta}}$ satisfy the Bianchi identities on $(\mathbb{C}^4, \mathcal{O}_{(1;2,6)})$.

By eliminating all λ-dependence, we have implicitly performed the push-forward of $\hat{\mathcal{A}}$ along π_1 onto $(\mathbb{C}^4, \mathcal{O}_{(1;2,6)})$. Let us define further tensor superfields, which could roughly be seen as extensions of the supercurvature fields and which capture the solutions to the above equations:

$$
\begin{aligned}
[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] &=: \varepsilon_{\dot{\alpha}\dot{\beta}} \mathcal{F}_{\alpha\beta} \,, \qquad [\nabla^i_{\dot{\alpha}\dot{\gamma}}, \nabla_{\beta\dot{\beta}}] =: \varepsilon_{\dot{\alpha}\dot{\beta}} \mathcal{F}^i_{\beta\dot{\gamma}} \,, \\
[\nabla^i_{\dot{\alpha}\dot{\gamma}}, \nabla^j_{\dot{\beta}\dot{\delta}}] &=: \varepsilon_{\dot{\alpha}\dot{\beta}} \mathcal{F}^{ij}_{\dot{\gamma}\dot{\delta}} \,,
\end{aligned} \tag{3.24}
$$

where $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{(\alpha\beta)}$ and $\mathcal{F}^{ij}_{\dot{\gamma}\dot{\delta}} = \mathcal{F}^{(ij)}_{(\dot{\gamma}\dot{\delta})} + \mathcal{F}^{[ij]}_{[\dot{\gamma}\dot{\delta}]}$. Note, however, that we introduced too many of these components. Considering the third equation in (3.23), one notes that for $i=j$,

[16] This assumption is crucial for the Penrose-Ward transform and reduces the space of possible $\mathcal{A}^{0,1}$ to an open subspace around $\mathcal{A}^{0,1}=0$.

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the terms symmetric in $\dot\gamma, \dot\delta$ vanish trivially. This means, that the components $\mathcal{F}^{ii}_{(\dot\gamma\dot\delta)}$ are in fact superfluous and we can ignore them in the following discussion. The second and third equations in (3.24) can be contracted with $\varepsilon^{\dot\alpha\dot\gamma}$ and $\varepsilon^{\dot\beta\dot\delta}$, respectively, which yields

$$-2\nabla_{\beta\dot\beta}\hat{\mathcal{A}}^i_{[\dot 1\dot 2]} = \mathcal{F}^i_{\beta\dot\beta} \qquad \text{and} \qquad -2\nabla^i_{\dot\alpha\dot\gamma}\hat{\mathcal{A}}^j_{[\dot 1\dot 2]} = \mathcal{F}^{ij}_{\dot\gamma\dot\alpha}\,. \tag{3.25}$$

Furthermore, using Bianchi identities, one obtains immediately the following equations:

$$\nabla^{\alpha\dot\beta}\mathcal{F}^i_{\alpha\dot\gamma} = 0 \qquad \text{and} \qquad \nabla_{\alpha\dot\alpha}\mathcal{F}^{ij}_{\dot\beta\dot\gamma} = \nabla^i_{\dot\alpha\dot\beta}\mathcal{F}^j_{\alpha\dot\gamma}\,. \tag{3.26}$$

Due to self-duality, the first equation is in fact equivalent to $\nabla^{\alpha\dot\beta}\nabla_{\alpha\dot\beta}\hat{\mathcal{A}}^i_{[\dot 1\dot 2]} = 0$, as is easily seen by performing all the spinor index sums. From the second equation, one obtains the field equation $\nabla_\alpha{}^{\dot\beta}\mathcal{F}^{(12)}_{\dot\beta\dot\gamma} = -2[\hat{\mathcal{A}}^{(1}_{[\dot 1\dot 2]}, \nabla_{\alpha\dot\gamma}\hat{\mathcal{A}}^{2)}_{[\dot 1\dot 2]}]$ after contracting with $\varepsilon^{\dot\alpha\dot\beta}$.

To analyze the actual field content of this theory, we choose *transverse gauge* as in [37], i.e. we demand

$$\eta^{\dot\alpha}_k\hat{\mathcal{A}}^k_{\dot\alpha} = 0\,. \tag{3.27}$$

This choice reduces the group of gauge transformations to ordinary, group-valued functions on the body of $\mathbb{C}^{4|8}$. By using the identities $\eta^{\dot\beta}_2\hat{\mathcal{A}}^1_{\dot\alpha\dot\beta} = \hat{\mathcal{A}}^1_{\dot\alpha}$ etc., one sees that the above transverse gauge is equivalent to the transverse gauge for the reduced structure sheaf:

$$y^{\dot\alpha\dot\beta}_i\hat{\mathcal{A}}^i_{\dot\alpha\dot\beta} = \eta_1^{(\dot\alpha}\eta_2^{\dot\beta)}\hat{\mathcal{A}}^1_{\dot\alpha\dot\beta} + \eta_3^{(\dot\alpha}\eta_4^{\dot\beta)}\hat{\mathcal{A}}^2_{\dot\alpha\dot\beta} = 0\,. \tag{3.28}$$

In the expansion in the ys, the lowest components of $\mathcal{F}_{\alpha\beta}$, $\hat{\mathcal{A}}^i_{[\dot 1\dot 2]}$ and $\mathcal{F}^{(12)}_{(\dot\alpha\dot\beta)}$ are the self-dual field strength $f_{\alpha\beta}$, two complex scalars ϕ^i and the auxiliary field $G_{\dot\alpha\dot\beta}$, respectively. The two scalars ϕ^i can be seen as remainders of the six scalars contained in the $\mathcal{N}=4$ SDYM multiplet, which will become even clearer in the real case. The remaining components $\hat{\mathcal{A}}^i_{(\dot\alpha\dot\beta)}$ vanish to zeroth order in the ys due to the choice of transverse gauge. The field $\mathcal{F}^i_{\alpha\dot\alpha}$ does not contain any new physical degrees of freedom, as seen from the first equation in (3.25), but it is a composite field. The same holds for $\mathcal{F}^{[12]}_{[\dot\gamma\dot\alpha]}$ as easily seen by contracting the second equation in (3.25) by $\varepsilon^{\dot\gamma\dot\alpha}$: $\mathcal{F}^{[12]}_{[\dot 1\dot 2]} = -2[\hat{\mathcal{A}}^1_{[\dot 1\dot 2]}, \hat{\mathcal{A}}^2_{[\dot 1\dot 2]}]$.

The superfield equations of motion (3.26) are in fact equivalent to the equations

$$f_{\dot\alpha\dot\beta} = 0\,, \qquad \Box\phi^i = 0\,, \qquad \varepsilon^{\dot\gamma\dot\alpha}\nabla_{\alpha\dot\alpha}G_{\dot\gamma\dot\delta} + 2[\phi^{(1}, \nabla_{\alpha\dot\delta}\phi^{2)}] = 0\,. \tag{3.29}$$

To lowest order in the ys, the equations obviously match. Higher orders in the ys can be verified by defining the Euler operator $D := y^{\dot\alpha\dot\beta}_i\nabla^i_{(\dot\alpha\dot\beta)} = y^{\dot\alpha\dot\beta}_i\partial^i_{(\dot\alpha\dot\beta)}$ and applying D on the superfields and equations of motion which then turn out to be satisfied, if the equations (3.29) are fulfilled.

3.4 The B-model on $\mathcal{P}^{3\oplus 1|0}$

3.4.1 Geometrical considerations

The discussion for $\mathcal{P}^{3\oplus 1|0}$ follows the same lines as for $\mathcal{P}^{3\oplus 2|0}$ and is even simpler. Consider again the supertwistor space $\mathcal{P}^{3|4} = (\mathcal{P}^3, \mathcal{O}_{[4]})$. This time, let us introduce the following differential operators:

$$\tilde{\mathcal{D}}^{kl}_\pm := \eta^\pm_k\frac{\partial}{\partial\eta^\pm_l} \qquad \text{for } k,l = 1,\dots,4\,, \tag{3.30}$$

which are maps $\mathcal{O}_{[4]} \to \mathcal{O}_{[4]}$. The space $\mathcal{P}^3$ together with the extended structure sheaf[17]

$$\mathcal{O}_{(1,1)} := \bigcap_{k\neq l} \ker \tilde{\mathcal{D}}^{kl}_+ = \bigcap_{k\neq l} \ker \tilde{\mathcal{D}}^{kl}_- \,, \tag{3.31}$$

which is a reduction of $\mathcal{O}_{[4]}$, is an order one thickening of $\mathcal{P}^3$, which we denote by $\mathcal{P}^{3\oplus 1|0}$. This manifold can be covered by two patches $\mathcal{U}_+$ and $\mathcal{U}_-$ on which we define the coordinates $(z^\alpha_\pm,\, \lambda_\pm,\, y^\pm := \eta^\pm_1\eta^\pm_2\eta^\pm_3\eta^\pm_4)$. The even nilpotent coordinate $y^\pm$ is a section of the line bundle $\mathcal{O}(4)$ with the identification $(y^\pm)^2 \sim 0$.

Similarly to the case $\mathcal{P}^{3\oplus 2|0}$, we have the following identities:

$$\begin{aligned}
\eta^\pm_2\eta^\pm_3\eta^\pm_4 \frac{\partial}{\partial y^\pm} &= \left.\frac{\partial}{\partial \eta^\pm_1}\right|_{\mathcal{O}_{(1,1)}} \,, \qquad & \eta^\pm_1\eta^\pm_3\eta^\pm_4 \frac{\partial}{\partial y^\pm} &= -\left.\frac{\partial}{\partial \eta^\pm_2}\right|_{\mathcal{O}_{(1,1)}} \,, \\
\eta^\pm_1\eta^\pm_2\eta^\pm_4 \frac{\partial}{\partial y^\pm} &= \left.\frac{\partial}{\partial \eta^\pm_3}\right|_{\mathcal{O}_{(1,1)}} \,, \qquad & \eta^\pm_1\eta^\pm_2\eta^\pm_3 \frac{\partial}{\partial y^\pm} &= -\left.\frac{\partial}{\partial \eta^\pm_4}\right|_{\mathcal{O}_{(1,1)}}
\end{aligned} \tag{3.32}$$

which lead to the formal identifications

$$\frac{\partial}{\partial y^\pm} = \frac{\partial}{\partial \eta^\pm_4}\frac{\partial}{\partial \eta^\pm_3}\frac{\partial}{\partial \eta^\pm_2}\frac{\partial}{\partial \eta^\pm_1} \qquad \text{and} \qquad \mathrm{d}y^\pm = \mathrm{d}\eta^\pm_4\mathrm{d}\eta^\pm_3\mathrm{d}\eta^\pm_2\mathrm{d}\eta^\pm_1 \,, \tag{3.33}$$

but again with a restriction of the Leibniz rule in formal manipulations of expressions written in the η-coordinates as discussed in the previous section.

The holomorphic sections of the bundle $\mathcal{P}^{3\oplus 1|0} \to \mathbb{C}P^1$ are defined by the equations

$$z^{\dot\alpha}_\pm = x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha} \qquad \text{and} \qquad y^\pm = y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}\lambda^\pm_{\dot\gamma}\lambda^\pm_{\dot\delta} \,. \tag{3.34}$$

From the obvious identification $y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)} = \eta_1^{(\dot\alpha}\eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta)}$ we see, that the product $y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}y^{(\dot\mu\dot\nu\dot\rho\dot\sigma)}$ will vanish, unless the number of indices equal to $\dot 1$ is the same as the number of indices equal to $\dot 2$. In this case, we have additionally the identity

$$\sum_p (-1)^{n_p} y^{p_1} y^{p_2} = 0 \,, \tag{3.35}$$

where p is a permutation of $\dot1\dot1\dot1\dot1\dot2\dot2\dot2\dot2$, p_1 and p_2 are the first and second four indices of p, respectively, and n_p is the number of exchanges of a $\dot 1$ and a $\dot 2$ between p_1 and p_2, e.g. $n_{\dot1\dot1\dot1\dot2\dot1\dot2\dot2\dot2} = 1$.

The more formal treatment is much simpler. We introduce the differential operators

$$\tilde{\mathcal{D}}^{klc} = \left(\eta^{\dot\alpha}_l\partial^l_{\dot\alpha} - \eta^{\dot\alpha}_k\partial^k_{\dot\alpha}\right) \quad \text{without summation over } k \text{ and } l \,, \tag{3.36}$$

$$\tilde{\mathcal{D}}^{kls} = \left(\partial^k_{\dot1}\partial^l_{\dot2} - \partial^k_{\dot2}\partial^l_{\dot1}\right) \,, \tag{3.37}$$

which map $\mathcal{O}_{[8]} \to \mathcal{O}_{[8]}$. Then the space $\mathbb{C}^4$ with the extended structure sheaf $\mathcal{O}_{(1;2,5)}$ obtained by reducing $\mathcal{O}_{[8]}$ to the overlap of kernels

$$\mathcal{O}_{(1;2,5)} := \bigcap_{k\neq l}\left(\ker \tilde{\mathcal{D}}^{klc} \cap \ker \tilde{\mathcal{D}}^{kls}\right) \tag{3.38}$$

[17] The same reduction can be obtained by imposing integral constraints [34].

is the moduli space described above. Thus, we have the following reduction of the full double fibration (3.3) for $\mathcal{N}=4$:

$$
\begin{tikzcd}[column sep=small]
& (\mathbb{C}^4\times\mathbb{C}P^1,\mathcal{O}_{[8]}\otimes\mathcal{O}_{\mathbb{C}P^1}) \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & & \longrightarrow & & (\mathbb{C}^4\times\mathbb{C}P^1,\mathcal{O}_{(1;2,5)}\otimes\mathcal{O}_{\mathbb{C}P^1}) \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
(\mathcal{P}^3,\mathcal{O}_{[4]}) & & (\mathbb{C}^4,\mathcal{O}_{[8]}) & & (\mathcal{P}^3,\mathcal{O}_{(1,1)}) & & (\mathbb{C}^4,\mathcal{O}_{(1;2,5)})
\end{tikzcd}
\tag{3.39}
$$

where $\mathcal{O}_{\mathbb{C}P^1}$ is again the structure sheaf of the Riemann sphere $\mathbb{C}P^1$. The tangent spaces along the leaves of the projection π_2 are spanned by the vector fields

$$
\begin{aligned}
V_\alpha^\pm &= \lambda_\pm^{\dot\alpha}\partial_{\alpha\dot\alpha}\,, \qquad & V_\alpha^\pm &= \lambda_\pm^{\dot\alpha}\partial_{\alpha\dot\alpha}\,,\\
D_\pm^k &= \lambda_\pm^{\dot\alpha}\frac{\partial}{\partial\eta_k^{\dot\alpha}}\,, \qquad & D^\pm_{\dot\beta\dot\gamma\dot\delta} &= \lambda_\pm^{\dot\alpha}\partial_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}
\end{aligned}
\tag{3.40}
$$

in the left and right double fibration in (3.39), where $k=1,\ldots,4$. The further identities

$$
\eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\frac{\partial}{\partial y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} = \left.\frac{\partial}{\partial\eta_1^{\dot\alpha}}\right|_{\mathcal{O}_{(1;2,5)}}\,, \qquad \eta_1^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\frac{\partial}{\partial y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} = -\left.\frac{\partial}{\partial\eta_2^{\dot\alpha}}\right|_{\mathcal{O}_{(1;2,5)}}\,, \tag{3.41}
$$

$$
\eta_1^{\dot\beta}\eta_2^{\dot\gamma}\eta_4^{\dot\delta}\frac{\partial}{\partial y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} = \left.\frac{\partial}{\partial\eta_3^{\dot\alpha}}\right|_{\mathcal{O}_{(1;2,5)}}\,, \qquad \eta_1^{\dot\beta}\eta_2^{\dot\gamma}\eta_3^{\dot\delta}\frac{\partial}{\partial y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} = -\left.\frac{\partial}{\partial\eta_4^{\dot\alpha}}\right|_{\mathcal{O}_{(1;2,5)}} \tag{3.42}
$$

are easily derived and from them it follows that e.g.

$$
D_\pm^1\big|_{\mathcal{O}_{(1;2,5)}} = \eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}D^\pm_{\dot\beta\dot\gamma\dot\delta} \qquad \text{and} \qquad D_\pm^2\big|_{\mathcal{O}_{(1;2,5)}} = -\eta_1^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}D^\pm_{\dot\beta\dot\gamma\dot\delta}\,. \tag{3.43}
$$

3.4.2 Field theoretical considerations

The topological B-model on $\mathcal{P}^{3\oplus1|0}$ is equivalent to hCS theory on $\mathcal{P}^{3\oplus1|0}$ and introducing a trivial rank n complex vector bundle $\mathcal{E}$ over $\mathcal{P}^{3\oplus1|0}$ with a connection $\mathcal{A}$, the action reads

$$
S = \int_{\mathcal{P}^{3\oplus1|0}_{\mathrm{ch}}} \Omega^{3\oplus1|0}\wedge\left(\mathcal{A}^{0,1}\wedge\bar\partial\mathcal{A}^{0,1}+\frac{2}{3}\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\right), \tag{3.44}
$$

with $\mathcal{P}^{3\oplus1|0}_{\mathrm{ch}}$ being the chiral subspace for which $\bar y^\pm=0$ and $\mathcal{A}^{0,1}$ the $(0,1)$-part of $\mathcal{A}$. The holomorphic volume form $\Omega^{3\oplus1|0}$ can be defined, e.g. on $\mathcal{U}_+$, as $\Omega^{3\oplus1|0}_+ = \mathrm{d}z_+^1\wedge\mathrm{d}z_+^2\wedge\mathrm{d}\lambda_+\wedge\mathrm{d}y^+$. Following exactly the same steps as in the case $\mathcal{P}^{3\oplus2|0}$, we again obtain the equations

$$
\begin{aligned}
\hat\psi_+V_\alpha^+\hat\psi_+^{-1} = \hat\psi_-V_\alpha^+\hat\psi_-^{-1} &=: \lambda_+^{\dot\alpha}\hat{\mathcal{A}}_{\alpha\dot\alpha} =: \hat{\mathcal{A}}_\alpha^+\,,\\
\hat\psi_+D_+^k\hat\psi_+^{-1} = \hat\psi_-D_+^k\hat\psi_-^{-1} &=: \lambda_+^{\dot\alpha}\hat{\mathcal{A}}_{\dot\alpha}^k =: \hat{\mathcal{A}}_+^k\,,\\
\hat\psi_+\partial_{\bar\lambda_+}\hat\psi_+^{-1} = \hat\psi_-\partial_{\bar\lambda_+}\hat\psi_-^{-1} &=: \hat{\mathcal{A}}_{\bar\lambda_+} = 0\,,\\
\hat\psi_+\partial_{\bar x^{\alpha\dot\alpha}}\hat\psi_+^{-1} = \hat\psi_-\partial_{\bar x^{\alpha\dot\alpha}}\hat\psi_-^{-1} &= 0\,.
\end{aligned}
\tag{3.45}
$$

and by considering the reduced structure sheaves, we can rewrite the second line this time as

$$
\eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\,\hat\psi_+D^+_{\dot\beta\dot\gamma\dot\delta}\hat\psi_+^{-1} = \eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\,\hat\psi_-D^+_{\dot\beta\dot\gamma\dot\delta}\hat\psi_-^{-1} =: \eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\,\lambda_+^{\dot\alpha}\hat{\mathcal{A}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} =: \eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\,\hat{\mathcal{A}}^+_{\dot\beta\dot\gamma\dot\delta} \tag{3.46}
$$

for $k = 1$ which yields $\eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\hat{\mathcal{A}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \hat{\mathcal{A}}^1_{\dot\alpha}$. Similar formulæ are obtained for the other values of k, with which one can determine the superfield expansion of $\hat{\mathcal{A}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$ again from the superfield expansion of $\hat{\mathcal{A}}^k_{\dot\alpha}$ by dropping the terms which are not in the kernel of the differential operators $\tilde{\mathcal{D}}^{klc}$ and $\tilde{\mathcal{D}}^{kls}$ for $k \neq l$.

Analogously to the case $\mathcal{P}^{3\oplus 2|0}$, one can rewrite the linear system behind (3.45), (3.46) for the reduced structure sheaf. For this, we define the covariant derivative $\nabla_{\alpha\dot\alpha} := \partial_{\alpha\dot\alpha} + [\hat{\mathcal{A}}_{\alpha\dot\alpha}, \cdot]$ and the first order differential operator $\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta} := \partial_{\dot\alpha\dot\beta\dot\gamma\dot\delta} + [\hat{\mathcal{A}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}, \cdot]$. Then we have

$$
\begin{aligned}
[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla_{\alpha\dot\beta}, \nabla_{\beta\dot\alpha}] &= 0\,, \\
\eta_k^{\dot\nu}\eta_m^{\dot\rho}\eta_n^{\dot\sigma}\left([\nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}, \nabla_{\alpha\dot\alpha}] + [\nabla_{\dot\alpha\dot\nu\dot\rho\dot\sigma}, \nabla_{\alpha\dot\mu}]\right) &= 0\,, \\
\eta_r^{\dot\beta}\eta_s^{\dot\gamma}\eta_t^{\dot\delta}\eta_k^{\dot\nu}\eta_m^{\dot\rho}\eta_n^{\dot\sigma}\Big([\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta}, \nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}] + [\nabla_{\dot\mu\dot\beta\dot\gamma\dot\delta}, \nabla_{\dot\alpha\dot\nu\dot\rho\dot\sigma}]\Big) &= 0\,,
\end{aligned} \tag{3.47}
$$

where (rst) and (kmn) are each a triple of pairwise different integers between 1 and 4. Again, in these equations the push-forward $\pi_{1*}\hat{\mathcal{A}}$ is already implied and solutions to (3.47) are captured by the following extensions of the supercurvature fields:

$$
\begin{aligned}
[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] &=: \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}\,, \\
[\nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}, \nabla_{\alpha\dot\alpha}] &=: \varepsilon_{\dot\alpha\dot\mu}\mathcal{F}_{\alpha\dot\nu\dot\rho\dot\sigma}\,, \\
[\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta}, \nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}] &=: \varepsilon_{\dot\alpha\dot\mu}\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\nu\dot\rho\dot\sigma}\,,
\end{aligned} \tag{3.48}
$$

where $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{(\alpha\beta)}$, $\mathcal{F}_{\alpha\dot\nu\dot\rho\dot\sigma} = \mathcal{F}_{\alpha(\dot\nu\dot\rho\dot\sigma)}$ and $\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\nu\dot\rho\dot\sigma} = \mathcal{F}_{(\dot\beta\dot\gamma\dot\delta)(\dot\nu\dot\rho\dot\sigma)}$ is symmetric under exchange of $(\dot\beta\dot\gamma\dot\delta) \leftrightarrow (\dot\nu\dot\rho\dot\sigma)$. Consider now the third equation of (3.47). Note that the triples (rst) and (kmn) will have two numbers in common, while exactly one is different. Without loss of generality, let $r \neq k$, $s = m$ and $t = n$. Then one easily sees, that the terms symmetric in $\dot\beta$, $\dot\nu$ vanish trivially. This means, that the field components $\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\nu\dot\rho\dot\sigma}$ which are symmetric in $\dot\beta$, $\dot\nu$ are again unconstrained additional fields, which do not represent any of the fields in the $\mathcal{N} = 4$ SDYM multiplet and we put them to zero, analogously to $\mathcal{F}^{ii}_{(\dot\gamma\dot\delta)}$ in the case $\mathcal{P}^{3\oplus 2|0}$.

The second equation in (3.48) can be contracted with $\varepsilon^{\dot\mu\dot\nu}$ which yields $2\nabla_{\alpha\dot\alpha}\hat{\mathcal{A}}_{[\dot 1\dot 2]\dot\rho\dot\sigma} = \mathcal{F}_{\alpha\dot\alpha\dot\rho\dot\sigma}$ and further contracting this equation with $\varepsilon^{\dot\alpha\dot\rho}$ we have $\nabla_{\alpha}{}^{\dot\alpha}\hat{\mathcal{A}}_{[\dot 1\dot 2]\dot\alpha\dot\sigma} = 0$. After contracting the third equation with $\varepsilon^{\dot\mu\dot\nu}$, one obtains

$$
-2\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\hat{\mathcal{A}}_{[\dot 1\dot 2]\dot\rho\dot\sigma} = \mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\alpha\dot\rho\dot\sigma}\,. \tag{3.49}
$$

The transversal gauge condition $\eta_k^{\dot\alpha}\hat{\mathcal{A}}^k_{\dot\alpha} = 0$ is on $\mathcal{O}_{(1;2,5)}$ equivalent to the condition

$$
y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\hat{\mathcal{A}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = 0\,, \tag{3.50}
$$

as expected analogously to the case $\mathcal{P}^{3\oplus 2|0}$. To lowest order in $y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$, $\mathcal{F}_{\alpha\beta}$ can be identified with the self-dual field strength $f_{\alpha\beta}$ and $\hat{\mathcal{A}}_{[\dot 1\dot 2]\dot\alpha\dot\beta}$ with the auxiliary field $G_{\dot\alpha\dot\beta}$. The remaining components of $\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\alpha\dot\rho\dot\sigma}$, i.e. those antisymmetric in $[\dot\alpha\dot\beta]$, are composite fields and do not contain any additional degrees of freedom which is easily seen by considering equation (3.49).

Applying the Euler operator in transverse gauge $D := y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\nabla_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)} = y^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\partial_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$, one can show that the lowest order field equations are equivalent to the full superfield equations of motion. Thus, (3.47) is equivalent to

$$f_{\dot\alpha\dot\beta} = 0 \qquad \text{and} \qquad \nabla^{\alpha\dot\alpha}G_{\dot\alpha\dot\beta} = 0\,. \tag{3.51}$$

Altogether, we found the compatibility condition for a linear system encoding purely bosonic SDYM theory *including* the auxiliary field $G_{\dot\alpha\dot\beta}$.

4. Fattened real manifolds

The field content of hCS theory on $\mathcal{P}^{3\oplus2|0}$ and $\mathcal{P}^{3\oplus1|0}$ becomes even more transparent after imposing a reality condition on these spaces. One can directly derive appropriate real structures from the one on $\mathcal{P}^{3|4}$, having in mind the picture of combining the Graßmann coordinates of $\mathcal{P}^{3|4}$ to the even nilpotent coordinates of $\mathcal{P}^{3\oplus2|0}$ and $\mathcal{P}^{3\oplus1|0}$. The real structure on $\mathcal{P}^{3|4}$ is discussed in detail in [4], one should note, however, that our normalization of fields is different from the one in this reference.

We have the following action of the two antilinear involutions τ_ε with $\varepsilon = \pm1$ on coordinates:

$$\tau_\varepsilon(z^1_+, z^2_+, \lambda_+) = \left(\frac{\bar z^2_+}{\bar\lambda_+}, \frac{\varepsilon\bar z^1_+}{\bar\lambda_+}, \frac{\varepsilon}{\bar\lambda_+}\right) \qquad \text{and} \qquad \tau_\varepsilon(z^1_-, z^2_-, \lambda_-) = \left(\frac{\varepsilon\bar z^2_-}{\bar\lambda_-}, \frac{\bar z^1_-}{\bar\lambda_-}, \frac{\varepsilon}{\bar\lambda_-}\right).$$

On $\mathcal{P}^{3\oplus2|0}$, we have additionally

$$\tau_\varepsilon(y^1_+, y^2_+) = \left(\frac{\bar y^1_+}{\bar\lambda^2_+}, \frac{\bar y^2_+}{\bar\lambda^2_+}\right) \qquad \text{and} \qquad \tau_\varepsilon(y^1_-, y^2_-) = \left(\frac{\bar y^1_-}{\bar\lambda^2_-}, \frac{\bar y^2_-}{\bar\lambda^2_-}\right), \tag{4.1}$$

and on $\mathcal{P}^{3\oplus1|0}$, it is

$$\tau_\varepsilon(y_+) = \frac{\bar y_+}{\bar\lambda^4_+} \qquad \text{and} \qquad \tau_\varepsilon(y_-) = \frac{\bar y_-}{\bar\lambda^4_-}\,. \tag{4.2}$$

In the formulation of the twistor correspondence, the coordinates $\lambda_\pm$ are usually kept complex for convenience sake [4]. We do the same while on all other coordinates, we impose the condition $\tau_\varepsilon(\cdot) = \cdot$. On the body of the moduli space, this will lead to a euclidean metric $(+,+,+,+)$ for $\varepsilon = -1$ and a Kleinian metric $(+,+,-,-)$ for $\varepsilon = +1$. Let us furthermore introduce the auxiliary functions[18]

$$\gamma_+ = \frac{1}{1-\varepsilon\lambda_+\bar\lambda_+} \qquad \text{and} \qquad \gamma_- = -\varepsilon\frac{1}{1-\varepsilon\lambda_-\bar\lambda_-}\,, \tag{4.3}$$

and the notation $(\hat\lambda^{\dot\alpha}) = (-\bar\lambda_{\dot1}, \varepsilon\bar\lambda_{\dot2})^T$.

The reality condition allows for the following identification:

$$\frac{\partial}{\partial\bar z^1_+} = \gamma_+ V^+_2 \qquad \text{and} \qquad \frac{\partial}{\partial\bar z^2_+} = \varepsilon\gamma_+ V^+_1\,, \tag{4.4}$$

[18] One should note that the $\gamma_\pm$ are not well-defined for $\varepsilon = 1$ on the whole of $\mathcal{P}^3$, but only on the subset for which $|\lambda| \neq 1$. Nevertheless, all the formulæ of the twistor correspondence can be used regardless of this fact. Therefore we ignore this subtlety in the following and refer to the discussion in [4, 17, 34].

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after which we can rewrite the hCS equations of motion, e.g. on $\mathcal{U}_+$, as

$$V^+_\alpha \hat{\mathcal{A}}^+_\beta - V^+_\beta \hat{\mathcal{A}}^+_\alpha + [\hat{\mathcal{A}}^+_\alpha, \hat{\mathcal{A}}^+_\beta] = 0\,, \tag{4.5}$$

$$\partial_{\bar{\lambda}_+} \hat{\mathcal{A}}^+_\alpha - V^+_\alpha \hat{\mathcal{A}}_{\bar{\lambda}_+} + [\hat{\mathcal{A}}_{\bar{\lambda}_+}, \hat{\mathcal{A}}^+_\alpha] = 0\,, \tag{4.6}$$

where the components of the gauge potential are defined via the contractions $\hat{\mathcal{A}}^\pm_\alpha := V^\pm_\alpha \lrcorner \mathcal{A}^{0,1}$, $\hat{\mathcal{A}}_{\bar{\lambda}_\pm} := \partial_{\bar{\lambda}_\pm} \lrcorner \mathcal{A}^{0,1}$, and we assumed a gauge for which $\hat{\mathcal{A}}^\pm_i := \partial_{\bar{y}^\pm_i} \lrcorner \mathcal{A}^{0,1} = 0$. On the space $\mathcal{P}^{3\oplus 2|0}$ together with the field expansion[19]

$$\hat{\mathcal{A}}^+_\alpha = \lambda^{\dot\alpha}_+ A_{\alpha\dot\alpha} + \gamma_+ y^+_i \hat{\lambda}^{\dot\alpha}_+ \phi^i_{\alpha\dot\alpha} + \gamma^3_+ y^+_1 y^+_2 \hat{\lambda}^{\dot\alpha}_+ \hat{\lambda}^{\dot\beta}_+ \hat{\lambda}^{\dot\gamma}_+ G_{\alpha\dot\alpha\dot\beta\dot\gamma}\,, \tag{4.7}$$

$$\hat{\mathcal{A}}_{\bar{\lambda}_+} = \gamma^2_+ y^+_i \phi^i - 2\varepsilon\, \gamma^4_+ y^+_1 y^+_2 \hat{\lambda}^{\dot\alpha}_+ \hat{\lambda}^{\dot\beta}_+ G_{\dot\alpha\dot\beta}\,, \tag{4.8}$$

the system of equations (4.5) and (4.6) is equivalent to (3.23). Furthermore, one can identify $\phi^i_{\alpha\dot\alpha} = -\frac{1}{2}\mathcal{F}^i_{\alpha\dot\alpha}$ and $G_{\alpha\dot\alpha\dot\beta\dot\gamma} = \frac{1}{6}\nabla^{(1}_{\dot\alpha(\dot\beta}\mathcal{F}^{2)}_{\alpha\dot\gamma)}$. On $\mathcal{P}^{3\oplus 1|0}$, we can use

$$\hat{\mathcal{A}}^+_\alpha = \lambda^{\dot\alpha}_+ A_{\alpha\dot\alpha} + \gamma^3_+ y^+ \hat{\lambda}^{\dot\alpha}_+ \hat{\lambda}^{\dot\beta}_+ \hat{\lambda}^{\dot\gamma}_+ G_{\alpha\dot\alpha\dot\beta\dot\gamma}\,, \tag{4.9}$$

$$\hat{\mathcal{A}}_{\bar{\lambda}_+} = \gamma^4_+ y^+ \hat{\lambda}^{\dot\alpha}_+ \hat{\lambda}^{\dot\beta}_+ G_{\dot\alpha\dot\beta}\,. \tag{4.10}$$

to have (4.5) and (4.6) equivalent with (3.47) and $G_{\alpha\dot\alpha\dot\beta\dot\gamma} = \frac{1}{6}\mathcal{F}_{\alpha\dot\alpha\dot\beta\dot\gamma}$.

For compactness of the discussion, we refrain from explicitly writing down all the reality conditions imposed on the component fields and refer again to [4] for further details.

One can reconstruct two action functionals, from which the equations of motion for the two cases arise. With our field normalizations, they read

$$S_{\mathcal{P}^{3\oplus 2|0}} = \int \mathrm{d}^4x\, \mathrm{tr}\left(G^{\dot\alpha\dot\beta} f_{\dot\alpha\dot\beta} - \phi^{(1}\Box\phi^{2)}\right), \tag{4.11}$$

$$S_{\mathcal{P}^{3\oplus 1|0}} = \int \mathrm{d}^4x\, \mathrm{tr}\left(G^{\dot\alpha\dot\beta} f_{\dot\alpha\dot\beta}\right). \tag{4.12}$$

The action $S_{\mathcal{P}^{3\oplus 1|0}}$ has first been proposed in [38].

5. Exotic supermanifolds and Yau's theorem

One can push the formalism of exotic supermanifolds a little bit further to proceed with an analysis similar to [21] and ask, whether (an appropriate extension of) Yau's theorem is valid for fattened complex manifolds which are Calabi-Yau, as the ones considered in this paper[20]. This theorem states that every Kähler manifold with vanishing first Chern class, or equivalently, with a globally defined holomorphic volume form, admits a Ricci-flat metric in every Kähler class.

[19] Note that this expansion is determined by the geometry of $\mathcal{P}^{3\oplus 2|0}$, cf. [4].

[20] For related work, see [39] and [40].

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We start from a $(k \oplus l|q)$-dimensional exotic supermanifold with local coordinate vector $(x^1, \dots, x^k, y^1, \dots, y^l, \zeta^1, \dots, \zeta^q)^T$. An element of the tangent space is described by a vector $(X^1, \dots, X^k, Y^1, \dots, Y^l, Z^1, \dots, Z^q)^T$. Both the metric and linear coordinate transformations on this space are defined by nonsingular matrices

$$K = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix}, \tag{5.1}$$

where the elements A, B, D, E, J are of even and G, H, C, F are of odd parity. As a definition for the *extended supertrace* of such matrices, we choose

$$\mathrm{etr}(K) := \mathrm{tr}(A) + \mathrm{tr}(E) - \mathrm{tr}(J)\,, \tag{5.2}$$

which is closely related to the supertrace and which is the appropriate choice to preserve cyclicity: $\mathrm{etr}(KM) = \mathrm{etr}(MK)$. Similarly to [35], we define the extended superdeterminant by

$$\delta \ln \mathrm{edet}(K) := \mathrm{etr}(K^{-1}\delta K) \qquad \text{together with } \mathrm{edet}(\mathbb{1}) := 1\,, \tag{5.3}$$

which guarantees $\mathrm{edet}(KM) = \mathrm{edet}(K)\mathrm{edet}(M)$. Proceeding analogously to [35], one decomposes K into the product of a lower triangular matrix, a block diagonal matrix and an upper diagonal matrix. The triangular matrices can be chosen to have only 1 as diagonal entries and thus do not contribute to the total determinant. The block diagonal matrix is of the form

$$K' = \begin{pmatrix} A & 0 & 0 \\ 0 & E - DA^{-1}B & 0 \\ 0 & 0 & R \end{pmatrix}, \tag{5.4}$$

with $R = J - GA^{-1}C + (H - GA^{-1}B)(E - DA^{-1}B)^{-1}(F - DA^{-1}C)$. The determinant of a block diagonal matrix is easily calculated and in this case we obtain

$$\mathrm{edet}(K) = \mathrm{edet}(K') = \frac{\det(A)\det(E - DA^{-1}B)}{\det(R)}\,. \tag{5.5}$$

Note that for the special case of no even nilpotent dimensions, for which one should formally set $B = D = F = G = 0$, one recovers the formulæ for the supertrace ($E = 0$) and the superdeterminant ($E = \mathbb{1}$ to drop the additional determinant).

In [21], the authors found that Kähler supermanifolds with one fermionic dimension admit Ricci-flat supermetrics if and only if the body of the Kähler supermanifold admits a metric with vanishing scalar curvature. Let us investigate the same issue for the case of an $(p \oplus 1|0)$-dimensional exotic supermanifold Y with one even nilpotent coordinate y. We denote the ordinary p-dimensional complex manifold embedded in Y by X. The extended Kähler potential on Y is given by a real-valued function $\mathcal{K} = f^0 + f^1 y\bar{y}$, such that the metric takes the form

$$g := \left(\partial_i \bar{\partial}_{\bar{\jmath}} \mathcal{K}\right) = \begin{pmatrix} f^0_{,i\bar{\jmath}} + f^1_{,i\bar{\jmath}} y\bar{y} & f^1_{,i}\, y \\ f^1_{,\bar{\jmath}}\, \bar{y} & f^1 \end{pmatrix}. \tag{5.6}$$

For the extended Ricci-tensor to vanish, the extended Kähler potential has to satisfy the Monge-Ampère equation $\operatorname{edet}(g) := \operatorname{edet}(\partial_i \bar{\partial}_{\bar{j}} \mathcal{K}) = 1$. In fact, we find

$$\begin{aligned}
\operatorname{edet}(g) &= \det\left(f^0_{,i\bar{j}} + f^1_{,i\bar{j}} y\bar{y}\right)\left(f^1 - f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n} y\bar{y}\right) \\
&= \det\left[\left(f^0_{,i\bar{j}} + f^1_{,i\bar{j}} y\bar{y}\right)\left(\sqrt[p]{f^1} - \frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n} y\bar{y}}{p(f^1)^{\frac{p-1}{p}}}\right)\right] \\
&= \det\left[f^0_{,i\bar{j}}\sqrt[p]{f^1} + \left(f^1_{,i\bar{j}}\sqrt[p]{f^1} - f^0_{,i\bar{j}}\frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n}}{p(f^1)^{\frac{p-1}{p}}}\right) y\bar{y}\right] \\
&= \det\left[f^0_{,l\bar{j}}\sqrt[p]{f^1}\right] \det\left[\delta_i^k + \left(g^{\bar{m}k} f^1_{,i\bar{m}} - \delta_i^k \frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n}}{p f^1}\right) y\bar{y}\right],
\end{aligned}$$

where $g^{\bar{m}n}$ is the inverse of $f^0_{,n\bar{m}}$. Using the relation $\ln\det(A) = \operatorname{tr}\ln(A)$, we obtain

$$\operatorname{edet}(g) = \det\left[f^0_{,l\bar{j}}\sqrt[a]{f^1}\right]\left(1 + \left(g^{\bar{m}i} f^1_{,i\bar{m}} - \frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n}}{f^1}\right) y\bar{y}\right). \tag{5.7}$$

From demanding extended Ricci-flatness, it follows that

$$f^1 = \frac{1}{\det\left(f^0_{,l\bar{j}}\right)} \qquad \text{and} \qquad \left(g^{\bar{j}i} f^1_{,i\bar{j}} - \frac{f^1_{\bar{j}} g^{\bar{j}i} f^1_{,i}}{f^1}\right) = 0\,. \tag{5.8}$$

The second equation can be simplified to

$$g^{\bar{j}i}\left(f^1_{,i\bar{j}} - \frac{f^1_{\bar{j}} f^1_{,i}}{f^1}\right) = f^1 g^{\bar{j}i}\left(\ln(f^1)\right)_{,i\bar{j}} = 0\,, \tag{5.9}$$

and together with the first equation in (5.8), it yields

$$g^{\bar{j}i}\left(\ln\frac{1}{\det\left(f^0_{,k\bar{j}}\right)}\right)_{,i\bar{j}} = -g^{\bar{j}i}\left(\ln\det\left(f^0_{,k\bar{j}}\right)\right)_{,i\bar{j}} = -g^{\bar{j}i} R_{,i\bar{j}} = 0\,. \tag{5.10}$$

This equation states that an exotic supermanifold Y of dimension $(p \oplus 1|0)$ admits an extended Ricci-flat metric if and only if the embedded ordinary manifold X has vanishing scalar curvature. A class of examples for which this additional condition is not satisfied are the weighted projective spaces $W\mathbb{C}P^{m-1\oplus 1|0}(1,\ldots,1\oplus m|\cdot)$ which have vanishing first Chern class but do not admit a Kähler metric with vanishing Ricci scalar.

Thus, we obtained exactly the same result as in [21], which is somewhat surprising as the definition of the extended determinant involved in our calculation strongly differs from the definition of the superdeterminant. However, this agreement might be an indication that fattened complex manifolds — together with the definitions made in this paper — fit nicely in the whole picture of extended Calabi-Yau spaces.

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